Matrix Theory
Exercise 1, Spring 2007

1. Let
$$A(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$
Prove that $A(\alpha+\beta) = A(\alpha)A(\beta)$. What matrix is $A(\alpha)A(-\alpha)$?

2. Let
$$A = \begin{bmatrix} a & x & y \\ 0 & b & z \\ 0 & 0 & c \end{bmatrix},$$
where $a, b, c \neq 0$. Calculate the inverse A^{-1} of A.

3. a) Show that $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$ for every $A, B \in \mathbb{C}_{n\times n}$
   b) Show that $(Ax|y) = (x|A^*y)$ for every $A \in \mathbb{C}_{n\times n}$ and $x, y \in \mathbb{C}^n$, where $(x|y)$ is the inner product in C^n.

4. Show that if A, B, and $A+B$ are nonsingular, then also $A^{-1}+B^{-1}$ is nonsingular and
$$(A+B)^{-1} = A^{-1} - A^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}.$$

5. Define a mapping $A\colon K^n \to K^n$ ($K = \mathbb{R}$ or $\mathbb{C}$) such that
$$A(x_1, x_2, \dots, x_n) = (x_1, x_2 - x_1, \dots, x_n - x_{n-1}).$$
Show that the mapping A is linear. What is the dimension $\dim \mathcal{R}(A)$ of the range of A?

6. Let $A\colon K^n \to K^n$ be a linear transformation. Show that the following conditions are equivalent:
   (i) A is injective;
   (ii) A is surjective;
   (iii) A is bijective.

7. Suppose that $V = \{x \mid x \text{ on kuvaus } \mathbb{R} \to \mathbb{R}\}$ and define a mapping $P\colon V \to V$ such that
$$Px(t) = \frac{1}{2}(x(t) + x(-t)), \quad \text{for every } x \in V, t \in \mathbb{R}.$$
Show that P is a projection. What is the direct sum $\mathcal{N}(P) \oplus \mathcal{R}(P)$?

8. Let $P, Q\colon V \to V$ be projections for which $\mathcal{N}(P) \subseteq \mathcal{N}(Q)$. Show that $QP = Q$.

Note. Problems 7 and 8 are point exercises.