Matrix Theory

Exercise 2, Spring 2007

1. Let $\overline{z} \in K^n$ be a solution for $A\overline{x} = \overline{c}$, where $A \in K_{n\times n}$. Show that

   (i) if $\overline{v} \in \mathcal{N}(A)$, then $\overline{z} + \overline{v}$ is also solution for the equation $A\overline{x} = \overline{c}$.

   (ii) for every solution $\overline{x} \in K^n$ there exists $\overline{v} \in \mathcal{N}(A)$ such that $\overline{x} = \overline{z} + \overline{v}$.

2. Show that the determinant of the matrix $A = \begin{bmatrix} B_1 & C \\ 0 & B_2 \end{bmatrix}$, where B_1 and B_2 are square matrices, is $\det A = \det B_1 \cdot \det B_2$.

   (**Hint.** Present A in the form $A = C_1C_2$, where $C_1 = \begin{bmatrix} I & 0 \\ 0 & B_2 \end{bmatrix}$.)

3. Suppose that $A \in K_{m\times n}$ and $B \in K_{n\times m}$. Show that

$$\det \begin{bmatrix} 0 & A \\ B & I \end{bmatrix} = \det(-AB) \quad (\text{i.e.} \ = (-1)^m \det(AB)).$$

   (**Hint.** Use the previous problem.)

4. So-called *Gram determinant* of the vectors $\overline{x}_1, \ldots, \overline{x}_k \in \mathbb{C}^n$ $(k \leq n)$ is $G(\overline{x}_1, \ldots, \overline{x}_k) = \det(A^*A)$, where $A = [\overline{x}_1, \ldots, \overline{x}_k]$ and $A^* = (\overline{A})^{\mathsf{T}}$. Show, by using Binet-Cauchy formula, that $G \geq 0$ always.

5. Suppose that $A = [a_{ij}]_{n\times n} \in K_{n\times n}$ is an upper triangular matric, where $a_{kk} \neq 0$ whenever $k = 1, 2, \ldots, n$. Show that the adjungate $\operatorname{adj} A$ and the inverse A^{-1} are upper triangular matrices.

6. Prove the following identity (so-called Cauchy identity):

$$\det \begin{bmatrix} a_1c_1 + \ldots + a_nc_n & a_1d_1 + \ldots + a_nd_n \\ b_1c_1 + \ldots + b_nc_n & b_1d_1 + \ldots + b_nd_n \end{bmatrix} = \sum_{1\leq i<j\leq n} \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} \begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix}.$$

   Show using the formula above that

$$(|a_1|^2 + \ldots + |a_n|^2)(|b_1|^2 + \ldots + |b_n|^2) \geq |(a_1\overline{b}_1 + \ldots + a_n\overline{b}_n)|^2$$

   for every $a_i, b_i \in \mathbb{C}$.

   (**Hint.** Present the left side as a product of two matrices and use Binet-Cauchy formula.)

Note. Problems 5 and 6 are point exercises.