Matrix Theory

Exercise 3, Spring 2007

1. Let $A \in K_{n\times n}$. Show that

    (i) if $r(A) = n$, then $r(\operatorname{adj} A) = n$;

    (ii) if $r(A) = n - 1$, then $r(\operatorname{adj} A) = 1$;

    Hint. Use the Sylvester's inequality $r(A \cdot \operatorname{adj} A) \geq r(A) + r(\operatorname{adj} A) - n$.

    (iii) if $r(A) < n - 1$, then $r(\operatorname{adj} A) = 0$.

2. Calculate the LU decomposition for the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{bmatrix}$$

    What is $\det A$?

3. Consider the matrix A in the preceding problem. Solve the equation $Ax = (1, 1, 1)^{\mathsf{T}}$ by using the LU decomposition of A.

4. Suppose that all the principal minors of $A \in K_{n\times n}$ (including $\det A$) are nonzero. Show that $A = LDU$, where D is an diagonal matrix, L and U are lower and upper triangular matrices, respectively, with diagonal entries equal to 1. What can you say about uniqueness of the decompostion?

5. Show that a matrix $A \in \mathbb{C}_{n\times n}$ is hermitian (i.e. $A^* = A$) if and only if $(Ax|y) = (x|Ay)$ for every $x, y \in \mathbb{C}^n$.

6. Let $A \in \mathbb{C}_{n\times n}$. Show that if $x^*Ax = 0$ for every $x \in \mathbb{C}^n$, then $A = 0 \in \mathbb{C}_{n\times n}$.

    Hint. Use the vectors e_i, $e_i + e_j$ and $e_i + \mathrm{i}e_j$.

7. A matrix $A \in \mathbb{C}_{n\times n}$ is called unitary if $A^* = A^{-1}$, i.e. $A^*A = I$. Show, by using problem 5, that A is unitary if and only if $(Ax|Ax) = (x|x)$ for every $x \in \mathbb{C}^n$.

Note. You may choose **two** of the problems 5, 6 and 6 as point exercises. By doing all three problems 5, 6 and 6, you can compensate one undone point problem.