Matrix Theory
Exercise 4, Spring 2007

1. Let $A\colon V \to V$ be a linear transformation. Show that if for some vector $x_0 \in V$ ($x_0 \neq \overline{0}$) we have
$$Ax_0 = \lambda x_0,$$
where $\lambda \in K$, then the subspace $S = \mathcal{L}\{x_0\}$ is A-invariant.

2. Calculate the characteristic polynomial of the matrix
$$A = \begin{bmatrix} 8 & 2 & -2 \\ 3 & 3 & -1 \\ 24 & 8 & -6 \end{bmatrix}$$
by using
   (i) $\det(\lambda I - A)$ and
   (ii) the principal minors of A.

   What is the spectrum $\sigma(A)$ of A and the corresponding eigenvectors.

3. Let $A \in \mathbb{C}_{n\times n}$. Show that the following are equivalent:
   (a) A is unitary;
   (b) $(Ax|Ay) = (x|y)$ for every $x, y \in \mathbb{C}^n$;
   (c) columns of A are ortonormal;
   (d) rows of A are ortonormal.

   (**Hint.** In (c) ja (d), look the (i,j)-entries in the matrices A^*A and AA^* by using inner product.)

4. Suppose that $A \in \mathbb{C}_{n\times n}$ is unitary. Show that $|\lambda| = 1$ for every eigenvalue $\lambda \in \mathbb{C}$ of A. Show that $|\det A| = 1$.

5. Show that if $A \in K_{n\times n}$ is hermitian (i.e. $A^* = A$) and positive definite (i.e. $x^*Ax > 0$ for every $x \in K^n \setminus \{\overline{0}\}$), then $\det A > 0$.

   (**Hint.** Take eigenvectors.)

6. Show that the eigenvalues of a hermitian matrix are real and eigenvectors corresponding to distinct eigenvalues are ortogonaaliset.

   (**Hint.** Show first that $\lambda = \overline{\lambda}$ for every eigenvalue λ.)

7. Let $T\colon V \to V$ be a linear transformation and $\dim V = n$. Suppose that $S \subseteq V$ is a T-invariant subspace for which $\dim S = r$. Show that then the matrix represenation A of T can be written as
$$A = \begin{bmatrix} A_1 & B \\ 0 & A_2 \end{bmatrix},$$
where $A_1 \in K_{r\times r}$ and $A_2 \in K_{(n-r)\times(n-r)}$.

   (**Hint.** Take V to be a direct sum $S \oplus S'$. Note that S' is not necessarily T-invariant.)

(All the vector spaces are assumed to be finite dimensional.)

Note. Problems 6 and 7 are point exercises.