Matrix Theory
Exercise 5, spring 2007

1. Show that eigenvectors of $A \in K_{n\times n}$ corresponding to distinct eigenvalues are linearly independent.

2. Find eigenvalues of $A^k + 3A + 2I$ $(k = 1, 2, \ldots)$ where
$$A = \begin{bmatrix} 8 & 2 & -2 \\ 3 & 3 & -1 \\ 24 & 8 & -6 \end{bmatrix}$$
(**Hint.** Find eigenvalues of A (see Exercise 4, problem 2.))

3. When 2×2-matrix is diagonalizable (i.e., simple in terms of Lancaster & Tismenentsky)?

4. Assume that matrices $A \in K_{n\times n}$ and $B \in K_{n\times n}$ are similar. Show that
   (a) $\det A = \det B$ and $r(A) = r(B)$;
   (b) $c_A(\lambda) = c_B(\lambda)$ and $\operatorname{tr} A = \operatorname{tr} B$;
   (c) A^{T} ja B^{T} are similar;
   (d) $p(A)$ ja $p(B)$ are similar whenever $p(\lambda)$ is skalar polynomial.

   Furthermore, show that matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are not similar although their ranks, determinants, characteristic polynomials and traces are equal.

5. Suppose that matrices A and B are diagonalizable. Show that A and B are similar if and only if $c_A(\lambda) = c_B(\lambda)$. (See previous problem.)

6. Let $A \in K_{n\times n}$. Show that the eigenvalues corresponding to the left eigenvectors of A are the same as the eigenvalues corresponding to right eigenvectors of A. (That is, we do not need to consider left and right eigenvalues.)

7. Show that the $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is diagonalizable and find the projections $G_j = x_j y_j^{\mathsf{T}}$ in the Spectral Theorem (Theorem 3.30). Calculate A^{20} using the Spectral Theorem.

8. Show that matrices AB and BA have the same characteristic polynomial whenever $A, B \in \mathbb{C}_{n\times n}$.

   (**Hint.** Use the equation
$$\underbrace{\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}}_{E} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}}_{F}$$
   and show that E and F are similar.)

Note. Problems 7 and 8 are point exercises.