Matrix Theory

Exercise 6, Spring 2007

1. Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

   Is A diagonalizable (i.e. simple)? Is A unitarily similar to some diagonal matrix? Is A normal?

2. Let $\lambda_1 = -1$, $\lambda_2 = 1$ ja $\lambda_3 = 0$ be eigenvalues of a matrix A and $x_1 = (-1, 1, 1)^{\mathbf{T}}$, $x_2 = (-1, 4, 1)^{\mathbf{T}}$ and $x_3 = (1, 2, 1)^{\mathbf{T}}$ their corresponding eigenvectors. Define the matrix A.

3. Let $A \in K_{n\times n}$ be an idempotent matrix (i.e. projection), that is, $A^2 = A$.

   (a) Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

   (b) If x is an eigenvector of A corresponding to ann eigenvalue λ, calculate $A^{200}x$.

4. Let $A \in \mathbb{C}_{n\times n}$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. Show that

   (a) A^*A is hermitian;

   (b) $\operatorname{tr}(A^*A) = \sum_{i,j} |a_{ij}|^2$;

   (c) $|\lambda_1|^2 + \ldots + |\lambda_n|^2 \leq \sum_{i,j} |a_{ij}|^2 = \operatorname{tr}(A^*A)$.

   (**Hint.** Use Schur Decomposition, i.e. the fact that every matrix is unitarily similar to some upper triangular matrix.)

5. Let $A = I - \alpha x x^*$, where $x \in \mathbb{C}^n \setminus \{\overline{0}\}$ and $\alpha = 2/\|x\|^2$ (reminder: $\|x\|^2 = (x|x) = x^*x$). Show that the matrix A is hermitian and unitary.

   Show that $\lambda = -1$ is an eigenvalue of A and the corresponding eigenvector is x.

6. Consider the matrix $A \in \mathbb{C}_{2\times 2}$ 1. Show that there exists a matrix B such that $B^2 = A$. Can the result be generalized for any diagonalizable (i.e. simple) matrix $A \in \mathbb{C}_{n\times n}$?

   (**Hint.** Use the decomposition $A = CDC^{-1}$ for diagonalizable matrices to define B.)

7. Let $A \in \mathbb{R}_{n\times n}$ be a symmetric and let r be its smallest and R its biggest eigenvalue. Show that

$$r \leq x^{\mathbf{T}}Ax \leq R$$

   whenever $x \in \mathbb{R}^n$ and $\|x\| = 1$. Formulate the result for hermitian matrix $A \in \mathbb{C}_{n\times n}$.

   (**Hint.** Use the fact that A has now n orthonormal eigenvectors.)

You may choose **two** of the problems 5, 6 and 6 as point exercises. By doing all three problems (5, 6 and 6), you can compensate one undone point problem.