Complex Analysis II

More Exercises

1. Calculate

   a) $\int_\gamma z^2 dz$, where γ is a semicircle from a to $-a$ ja $a > 0$, (in upper half-plane),

   b) $\int_\gamma \sin z \; dz$, where γ is a straight line from 0 to πi,

   c) $\int_{\partial D_1(1)} \overline{z}^2 dz$, d) $\int_{\partial D_4(1)} \frac{e^{3z}}{z - \pi i} dz$,

   e) $\int_{\partial D_3(0)} \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)} dz$, f) $\int_{\partial D_2(0)} \frac{e^{2z}}{(z+1)^4} dz$

   g) $\int_{\partial D_1(0)} \frac{\sin^6 z}{(z - \frac{\pi}{6})^3} dz$, h) $\frac{1}{2\pi} \int_0^{2\pi} \sin^2(\frac{\pi}{6} + 2e^{i\theta}) d\theta$.

   Ans. a) $-\frac{2a^3}{3}$, b) $1 - \cosh \pi$, c) $4\pi i$, d) $-2\pi i$, e) $4\pi i$, f) $\frac{8\pi i e^{-2}}{3}$, g) $\frac{21\pi i}{16}$, h) $\frac{1}{4}$.

2. Find the discs of convergence:

   a) $\sum_{n=0}^{\infty} \frac{(z+i)^n}{(n+1)(n+2)}$, b) $\sum_{n=0}^{\infty} \frac{1}{n^2 \cdot 3^n} (z+1)^n$, c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n$, d) $\sum_{n=0}^{\infty} \frac{z^n}{3^n + 1}$.

   Ans. a) $D_1(-i)$, b) $D_3(-1)$, c) $\mathbb{C}$, d) $D_3(0)$,

3. Calculate the following sums when $|z| < 1$:

   $$\text{a)} \sum_{n=1}^{\infty} n z^n, \quad \text{b)} \sum_{n=1}^{\infty} n^2 z^n.$$

   Ans: a) $\frac{z}{(1-z)^2}$, b) $\frac{z(1+z)}{(1-z)^3}$.

4. Find the Taylor series of

   $$\text{a)} \cos z, \; \text{b)} \sinh z, \; \text{c)} \frac{z}{(z^2+1)^2}$$

   in the origin.
   Ans. a) $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...$, b) $z + \frac{z^3}{3!} + \frac{z^5}{5!} - ...$, c) $z - 2z^3 + 3z^5 - ....$

5. Calculate

a) $\displaystyle\int_{\partial D_3(0)} \frac{dz}{(z-1)(z-2)^2}$, b) $\displaystyle\int_{\partial D_3(0)} \frac{dz}{z^4+4z^3}$, c) $\displaystyle\int_{\partial D_1(\frac{\pi}{2})} \tan z \; dz$,

d) $\displaystyle\int_{\partial D_{\frac{2\pi}{3}}(0)} \frac{dz}{z\sin z}$, e) $\displaystyle\int_{\partial D_1(0)} e^{\frac{1}{z^n}} dz,\ n \in \mathbb{N}$, f) $\displaystyle\frac{1}{2\pi i}\int_{\partial D_3(0)} \frac{e^z}{z^2(z^2+2z+2)} dz$.

Ans. a) 0, b) $\frac{\pi i}{32}$, c) $-2\pi i$, d) 0, e) $2\pi i$, if $n=1$; otherwise 0, f) $\frac{\cos 1}{2e}$.

6. Calculate

a) $\displaystyle\int_{\partial D_5(0)} \frac{e^z}{\cosh z} dz$, b) $\displaystyle\int_{\partial D_1(0)} e^{-1/z} \sin(1/z) dz$,

c) $\displaystyle\int_\gamma \frac{2+3\sin \pi z}{z(z-1)^2} dz$, where γ is the boundary of the square with vertices at $3+3i,\ 3-3i,\ -3+3i$ and $-3-3i$.

d) $\displaystyle\frac{1}{2\pi i}\int_\gamma \frac{e^{zt}}{z(z^2+1)} dz,\ t>0$, where γ is the boundary of the square with vertices at $1+i,\ -1+i,\ -1-i$ ja $1-i$.

Ans. a) $8\pi i$, b) $2\pi i$, c) $-6\pi i$, d) $1-\cos t$.

7. Calculate

a) $\displaystyle\int_0^\infty \frac{dx}{1+x^4}$, b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)^2}$, c) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+4x+5)^2}$,

Ans. a) $\frac{\pi}{2\sqrt{2}}$, b) $\frac{5\pi}{288}$, c) $\frac{\pi}{2}$.

8. Calculate

a) $\displaystyle\int_0^{2\pi} \frac{\sin 3t}{5-3\cos t} dt$, b) $\displaystyle\int_0^{2\pi} \frac{\cos 2t}{13-12\cos t} dt$, c) $\displaystyle\int_0^{2\pi} \frac{\cos 3t}{(5-3\cos t)^4} dt$,

Ans. a) 0, b) $\frac{8\pi}{45}$, c) $\frac{135\pi}{16384}$.

9. Calculate

a) $\displaystyle\int_{-\infty}^\infty \frac{(x+2)\sin 3x}{x^2+1} dx$, b) $\displaystyle\int_{-\infty}^\infty \frac{\cos(x/\sqrt{2})}{x^4+\pi^4} dx$, c) $\displaystyle\int_{-\infty}^\infty \frac{x\sin \pi x}{x^2+2x+5} dx$,

Ans. a) πe^{-3}, b) $e^{-\frac{\pi}{2}}/(\pi^2\sqrt{2})$, c) $-\pi e^{-2\pi}$.