Complex Analysis II

Exercise 4, Spring 2011

1. Suppose f is a function analytic in $\mathbb{C}$ such that
$$|f(z)| \leqslant \left|\frac{1+z}{1-z}\right|$$
for every $z \in \mathbb{C}$. Show that f is a constant function.

2. Let γ be a piecewise smooth curve in $\mathbb{C}$. Suppose (f_n) is a sequence of continuous functions on γ such that $f_n \to f$ uniformly on γ when $n \to \infty$. Show that
$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz.$$

3. Does the sequence $(f_n)_{n=1}^\infty$ converge on the set $E \subset \mathbb{C}$, when
$$\text{a)} \;\; f_n(z) = \frac{nz}{z+n}, \;\; E = D_1(0),$$
$$\text{b)} \;\; f_n(z) = \frac{nz}{nz+1}, \;\; E = \{z \in \mathbb{C} \;:\; |z| > 1\},$$
$$\text{c)} f_n(z) = \sum_{k=0}^{n} z^k, \;\; E = D_1(0).$$
Is the convergence uniform on E ?

4. Study whether the following series converge:
$$\text{a)} \;\; \sum_{n=1}^\infty \frac{e^{in}}{n^2}, \;\; \text{b)} \;\; \sum_{n=1}^\infty \frac{i^n}{n} \;\; \text{c)} \;\; \sum_{n=1}^\infty \frac{\sin(n+i)}{n^2} \qquad \text{d)} \;\; \sum_{n=1}^\infty n \sin\left(\frac{i}{n}\right).$$

5. For which $z \in \mathbb{C}$ the following series converge:
$$\text{a)} \;\; \sum_{n=1}^\infty \frac{1}{n+|z|}, \;\; \text{b)} \;\; \sum_{n=1}^\infty \frac{(-1)^n}{n+z}, \;\; \text{c)} \;\; \sum_{n=1}^\infty \frac{z^n}{1-z^n}.$$

6. Find the radius of convergence of the series
$$\sum_{n=0}^\infty \left(\frac{1}{1-\frac{1}{2}i}\right)^{n+1} (z-\frac{1}{2}i)^n$$
and calculate the sum.