Complex analysis II

Exercise 5, Spring 2011

1. Find the radii and disks of convergence for the following series:

$$\text{a)}\ \sum_{n=0}^{\infty} \frac{1}{2^n+1} z^n, \quad \text{b)}\ \sum_{n=0}^{\infty} \frac{1}{n^2}(z+1)^n, \quad \text{c)}\ \sum_{n=0}^{\infty} n^2 (z-\pi)^n, \quad \text{d)}\ \sum_{n=0}^{\infty} \frac{n^3}{3^n}(z-e)^n.$$

2. Let f be a continuous function on a domain A and $D = D_R(z_0)$ $(R > 0)$ an open disk such that $cl(D) \subset A$. Show that for all $z \in D$,

$$\lim_{n\to\infty} \int_{\partial D} \frac{f(w)(z-z_0)^{n+1}}{(w-z)(w-z_0)^{n+1}} dw = 0.$$

   Hint: Exercise 4.2.

3. Find the Taylor series of the following functions on given points and determine the radii of convergence.

$$\text{a)}\ f(z) = e^z,\ z_0 = \pi, \qquad \text{b)}\ f(z) = \sin z,\ z_0 = 0,$$
$$\text{c)}\ f(z) = \frac{1}{az+b},\ z_0 = 0, \quad \text{d)}\ f(z) = \frac{z}{z^2 - 2z + 5},\ z_0 = 1.$$

   $(a, b \in \mathbb{C} \setminus \{0\})$.

4. Determine the Taylor series of the functions $\sin z$ and $\cos z$ on $\frac{\pi}{4}$.

5. Study whether the following derivatives exist and if they do, calculate them:

$$\text{a)}\ f^{(9)}(2), \text{ when } f(z) = \sum_{n=0}^{\infty} \left(n \sin \frac{1}{3^n n}\right)^n (z-2)^n,$$
$$\text{b)}\ f^{(24)}(0), \text{ when } f(z) = \frac{z^{15}}{(3-z)^3}.$$

6. Show that the function $f(z) = \frac{z}{\sin z}$ is analytic on the origin and determine the first three terms of its Taylor series.