Complex analysis II

Exercise 6, Spring 2011

1. Let h be an analytic function on a domain A. Suppose that a point $a \in A$ is an accumulation point of the set $E := \{w \in A \ : \ h(w) = 0\}$. On lectures, it was shown that then there exists an $r > 0$ such that $D_r(a) \subset E$.
   Let $z_0 \in A$ be some accumulation point of E and $\gamma = \{\ \lambda(t) \ : \ t \in [0,1]\ \} \subset A$ be a continuous curve (that is, λ is continuous), with initial point z_0. Put
   $$T := \{s \in [0,1] \ : \ \lambda(t) \in E \text{ for every } t \in [0,s]\}.$$
   Further, denote $\tau := \sup T$. Show that
   i) $T \neq \varnothing$ and if $s \in T$ then $\lambda(s)$ is an accumulation point of the E .
   ii) $\tau \in T$
   iii) $\tau = 1$.
   Apply this to show that if E has an accumulation point in A, then $h(z) = 0 \ \forall z \in A$.
   Hint: Draw an image!

2. Let f be an analytic function on $D_R(0),\ R > 0$. Suppose further that f is not a constant function. Define $g :]0, R[\to \mathbb{R}$,
   $$g(r) = \max_{z \in cl(D_r(0))} |f(z)|.$$
   Show that g is strictly increasing.

3. Let f be an analytic function on a domain A such that the function $|f|$ has a local minimum on $z_0 \in A$ and $|f(z_0)| > 0$. Show that f is a constant function.

4. Let
   $$f(z) = \frac{\sum_{k=1}^{\infty} k(z-\pi)^{k-1}}{z}.$$
   Find the most general analytic extension of f.

5. Let
   $$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{2^{k+1}} \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} \frac{(z-i)^k}{(2-i)^{k+1}}.$$
   Show that g is an analytic extension of f.

6. Study, when the Laurent-series
   $$\sum_{n=-\infty}^{\infty} 2^{-|n|}(z+1)^n$$
   converges and calculate the sum.