MPTT2, viikko 12

(25) osittaisintegrointi: $\int f' g\,dx = fg - \int fg'\,dx + c$

a) $\int x e^{-x}\,dx$

$g(x) = x$ $\quad$ $f'(x) = e^{-x}$

$g'(x) = 1$ $\quad$ $f(x) = -\int -e^{-x}\,dx = -e^{-x}$

$= -e^{-x}\cdot x - \int -e^{-x}\cdot 1\,dx$

$= -xe^{-x} - e^{-x} + C$

$= -e^{-x}(x+1) + C$

b) $\int x^7 \ln x\,dx$

$g(x) = \ln x$ $\quad$ $f'(x) = x^7$

$g'(x) = \frac{1}{x}$ $\quad$ $f(x) = \int x^7\,dx = \frac{1}{8}x^8$

$= \frac{1}{8}x^8\cdot \ln x - \int \frac{1}{8}x^{\cancel{8}\,7}\cdot\frac{1}{\cancel{x}}\,dx$

$= \frac{1}{8}x^8\cdot\ln x - \int \frac{1}{8}x^7\,dx$

$= \frac{1}{8}x^8 \ln x - \frac{1}{8}\cdot\frac{1}{8}x^8 + C$

$= \frac{1}{8}x^8\left(\ln x - \frac{1}{8}\right) + C$

c) $\int x(-x+2)^5\,dx$

$g(x) = x$ $\quad$ $f'(x) = (-x+2)^5$

$g'(x) = 1$ $\quad$ $f(x) = -\int -(-x+2)^5\,dx = -\frac{1}{6}(-x+2)^6$

$= -\frac{1}{6}(-x+2)^6\cdot x - \int -\frac{1}{6}(-x+2)^6\cdot 1\,dx$

$= \frac{-x}{6}(-x+2)^6 - \frac{1}{6}\int -(-x+2)^6\,dx$

$= -\frac{x}{6}(-x+2)^6 - \frac{1}{6}\cdot\frac{1}{7}(-x+2)^7 + C$

$= -\frac{1}{6}(-x+2)^6\left(x + \frac{1}{7}(-x+2)\right) + C$

d) $\int \ln x\cdot\frac{1}{x}\,dx$

$g(x) = \ln x$ $\quad$ $f'(x) = \frac{1}{x}$

$g'(x) = \frac{1}{x}$ $\quad$ $f(x) = \int\frac{1}{x}\,dx = \ln|x|,\ x>0$ $= \ln x$

$= \ln x\cdot\ln x - \int \ln x\cdot\frac{1}{x}\,dx$

$= (\ln x)^2 - \int \ln x\cdot\frac{1}{x}\,dx$

$\Leftrightarrow 2\int \ln x\cdot\frac{1}{x}\,dx = (\ln x)^2 \quad \|:2$

$\Leftrightarrow \int \ln x\cdot\frac{1}{x}\,dx = \frac{(\ln x)^2}{2} + C$

26. a) $\int \frac{x^2}{x^3 - 3x^2 + 3x - 1}\,dx$

Jaetaan nimittäjä tekijöihin

$x^3 - 3x^2 + 3x - 1 = 0$

mahdolliset nollakohdat: $p: \pm 1,\ q: \pm 1 \qquad \frac{p}{q}: \pm 1$

$x = 1 \qquad 1^3 - 3 \cdot 1^2 + 3 \cdot 1 - 1 = 0$

$0 = 0$

$\Rightarrow x = 1$ eräs nollakohta

$\Rightarrow x - 1$ eräs tekijä

$$
\begin{array}{rl}
 & x^2 - 2x + 1 \\
x - 1 \,\big) & x^3 - 3x^2 + 3x - 1 \\
 & \underline{-(x^3 - x^2)} \\
 & -2x^2 + 3x \\
 & \underline{-(-2x^2 + 2x)} \\
 & x - 1 \\
 & \underline{-(x - 1)} \\
 & 0
\end{array}
$$

$\Rightarrow x^3 - 3x^2 + 3x - 1 = (x-1)(x^2 - 2x + 1)$

$x^2 - 2x + 1 = 0$

$x = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot 1}}{2} = \frac{2 \pm 0}{2} = 1$ (kaksinkertainen juuri)

$\Rightarrow x^3 - 3x^2 + 3x - 1 = (x-1)(x-1)^2 = (x-1)^3$

Osamurtokehitelmä:

$$\frac{x^2}{x^3 - 3x^2 + 3x - 1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \quad \| \cdot (x-1)^3$$

$$
\begin{aligned}
\Rightarrow x^2 &= A(x-1)^2 + B(x-1) + C \\
&= A(x^2 - 2x + 1) + Bx - B + C \\
&= Ax^2 - 2Ax + A + Bx - B + C \\
&= Ax^2 + (-2A + B)x + (A - B + C)
\end{aligned}
$$

$$\Rightarrow \begin{cases} A = 1 \\ -2A + B = 0 \\ A - B + C = 0 \end{cases} \Leftrightarrow \begin{cases} A = 1 \\ B = 2A = 2 \\ C = B - A = 1 \end{cases}$$

$$\Rightarrow \frac{x^2}{x^3-3x^2+3x-1} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}$$

$$\Rightarrow \int \frac{x^2}{x^3-3x^2+3x-1}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx + \int \frac{1}{(x-1)^3}\,dx$$

$$= \int \frac{1}{x-1}\,dx + 2\int (x-1)^{-2}\,dx + \int (x-1)^{-3}\,dx$$

$$= \ln|x-1| + 2\,\frac{(x-1)^{-1}}{-1} + \left(-\frac{1}{2}\right)(x-1)^{-2} + C$$

$$= \ln|x-1| - \frac{2}{x-1} - \frac{1}{2(x-1)^2} + C$$

b) $\displaystyle\int \frac{1}{x^5+2x^3+x}\,dx$

Jaetaan nimittäjä tekijöihin:

$x^5+2x^3+x=0$

$\Leftrightarrow \quad x(x^4+2x^2+1)=0$

$x^4+2x^2+1=0$

mahdolliset nollakohdat: $p: \pm 1$, $q: \pm 1$, $\frac{p}{q}: \pm 1$

$x=1$: $\quad 1^4+2\cdot 1^2+1=0$

$4=0$ epätosi $\Rightarrow$ ei toteuta

$x=-1$: $\quad (-1)^4+2\cdot(-1)^2+1=0$

$4=0$ epätosi $\Rightarrow$ ei rationaalisia nollakohtia.

Nyt $x^4 + 2x^2 + 1 = (x^2)^2 + 2(x^2) + 1 = 0$

$$x^2 = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 1}}{2} = \frac{-2 \pm 0}{2} = -1 \quad \text{(kaksinkert. juuri)}$$

$$\Rightarrow \quad x^4 + 2x + 1 = (x^2+1)^2$$
$$\Rightarrow \quad x^5 + 2x^3 + x = x(x^2+1)^2$$

osamurtokehitelmä

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \quad \| \cdot x(x^2+1)^2$$

$$\begin{aligned} \Rightarrow \quad 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4 + 2x^2 + 1) + (Bx^2 + Cx)(x^2+1) + Dx^2 + Ex \\ &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A + B + D)x^2 + (C+E)x + A \end{aligned}$$

$$\Rightarrow \begin{cases} A+B=0 \\ C=0 \\ 2A+B+D=0 \\ C+E=0 \\ A=1 \end{cases} \iff \begin{cases} B = -A = -1 \\ C=0 \\ D = -2A - B = -1 \\ E = -C = 0 \\ A = 1 \end{cases}$$

$$\Rightarrow \quad \frac{1}{x(x^2+1)^2} = \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}$$

$$\Rightarrow \int \frac{1}{x(x^2+1)^2}\,dx = \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int 2x(x^2+1)^{-2}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| - \frac{1}{2} \cdot \frac{(x^2+1)^{-1}}{-1} + C$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$$