MPTT 2 vko 13-14

27 a) $\int \frac{1}{1+\sqrt[3]{x+3}}\,dx$

sij. $t = \sqrt[3]{x+3} \quad \| (\,)^3$

$\Leftrightarrow t^3 = x+3$

$\Leftrightarrow x = t^3 - 3 = g(t)$

$\Leftrightarrow dx = 3t^2 = g'(t)\,dt$

$= \int \frac{1}{1+t} \cdot 3t^2\,dt$

$= 3\int \frac{t^2}{1+t}\,dt$

$$\begin{array}{r|l} & t-1 \\ \hline t+1 & t^2 \\ & -t^2 \mp t \\ \hline & -t \\ & \pm t \pm 1 \\ \hline & 1 \end{array}$$

$= 3\int \left(t-1 + \frac{1}{t+1} \right) dt$

$= 3\left(\frac{t^2}{2} - t + \ln|t+1| \right) + C$

$= \frac{3}{2}t^2 - 3t + 3\ln|t+1| + C$ TAKAISINSIJOITUS!

$= \frac{3}{2}\left(\sqrt[3]{x+3}\right)^2 - 3\sqrt[3]{x+3} + 3\ln\left|\sqrt[3]{x+3}+1\right| + C$

b) $\int \frac{1}{x\sqrt{2x-x^2}}\,dx$

sij. $x = \frac{1}{t} = t^{-1} = g(t) \Rightarrow t = \frac{1}{x}$

$dx = -t^{-2}\,dt = g'(t)\,dt$

$= \int \frac{-t^{-2}}{\frac{1}{t}\sqrt{2\cdot\frac{1}{t} - \left(\frac{1}{t}\right)^2}}\,dt$

$= \int \frac{-1}{t^2\cdot\frac{1}{t}\sqrt{\frac{2}{t} - \left(\frac{1}{t}\right)^2}}\,dt = \int \frac{-1}{t\sqrt{\frac{2}{t} - \left(\frac{1}{t}\right)^2}}\,dt$

$= \int \frac{-1}{\sqrt{t^2\left(\frac{2}{t} - \frac{1}{t^2}\right)}}\,dt = \int \frac{-1}{\sqrt{2t-1}}\,dt$

$= -\frac{1}{2}\int 2(2t-1)^{-\frac{1}{2}}\,dt$

$= -\frac{1}{2}\,\frac{(2t-1)^{\frac{1}{2}}}{\frac{1}{2}} + C = -\sqrt{2t-1} + C$ TAKAISIN SIJOITUS!

$= -\sqrt{\frac{2}{x} - 1} + C$

c) $\int \frac{x^2}{(1+2x)^{3/2}}\,dx$

sij. $1+2x = t^2 \Rightarrow t = \sqrt{1-2x}$

$\Leftrightarrow \quad 2x = t^2 - 1$

$\Leftrightarrow \quad x = \frac{1}{2}(t^2-1)$

$dx = t\,dt = g'(t)\,dt$

$$= \int \frac{\left(\frac{1}{2}(t^2-1)\right)^2}{(t^2)^{3/2}}\, t\,dt$$

$$= \int \frac{\frac{1}{4}(t^4-2t^2+1)}{t^3}\, t\,dt$$

$$= \frac{1}{4}\int \frac{t^4-2t^2+1}{t^2}\,dt = \frac{1}{4}\int (t^2 - 2 + t^{-2})\,dt$$

$$= \frac{1}{4}\left(\frac{t^3}{3} - 2t + \frac{t^{-1}}{-1}\right) + C = \frac{1}{4}\left(\frac{t^3}{3} - 2t - \frac{1}{t}\right) + C$$

TAKAISIN SIJOITUS! $= \frac{1}{4}\left(\frac{(\sqrt{1+2x})^3}{3} - 2\sqrt{1+2x} - \frac{1}{\sqrt{1+2x}}\right) + C$

d) $\int \frac{x^{\frac{1}{2}} + x^{\frac{1}{6}}}{x^{\frac{3}{4}}}\,dx$

sij. $x = t^{12} = g(t) \Rightarrow t = x^{\frac{1}{12}}$

$dx = 12t^{11}\,dt = g'(t)\,dt$

$$= \int \frac{(t^{12})^{\frac{1}{2}} + (t^{12})^{\frac{1}{6}}}{(t^{12})^{3/4}} \cdot 12t^{11}\,dt$$

$$= \int \frac{t^6 + t^2}{t^9} \cdot 12t^{11}\,dt$$

$$= 12\int (t^6 + t^2)\cdot t^2\,dt = 12\int (t^8 + t^4)\,dt$$

$$= 12\left(\frac{t^9}{9} + \frac{t^5}{5}\right) + C$$

$$= \frac{4}{3}t^9 + \frac{12}{5}t^5 + C$$

TAKAISIN SIJOITUS!

$$= \frac{4}{3}\left(x^{\frac{1}{12}}\right)^9 + \frac{12}{5}\left(x^{\frac{1}{12}}\right)^5 + C$$

$$= \frac{4}{3}x^{\frac{3}{4}} + \frac{12}{5}x^{\frac{5}{12}} + C$$

28 a) $\int_{-2}^{2}(x-1)^2\,dx$

Tapa 1: $\int_{-2}^{2}(x-1)^2\,dx = \int_{-2}^{2}(x^2-2x+1)\,dx$

$= \Big/_{-2}^{2}\left(\frac{x^3}{3}-\frac{\cancel{2}x^2}{\cancel{2}}+x\right)$

$= \frac{2^3}{3}-2^2+2-\left(\frac{(-2)^3}{3}-(-2)^2+(-2)\right)$

$= \frac{8}{3}-\cancel{4}+2+\frac{8}{3}+\cancel{4}+2 = \frac{28}{3}$

Tapa 2: $\int_{-2}^{2}(x-1)^2\,dx = \Big/_{-2}^{2}\frac{(x-1)^3}{3}$

$= \frac{(2-1)^3}{3}-\frac{((-2)-1)^3}{3}$

$= \frac{1^3}{3}-\frac{(-3)^3}{3} = \frac{28}{3}$

b) $\int_{1}^{2}\frac{2x+3}{x^2+3x+2}\,dx$ $\quad\parallel D(x^2+3x+2) = 2x+3$

$= \Big/_{1}^{2}\ln|x^2+3x+2|$

$= \ln|2^2+3\cdot 2+2| - \ln|1^2+3\cdot 1+2|$

$= \ln 12 - \ln 6 = \ln\left(\frac{12}{6}\right) = \ln 2$

c) $\int_{-2}^{2}x(x-1)^2\,dx = \int_{-2}^{2}x(x^2-2x+1)\,dx$

$= \int(x^3-2x^2+x)\,dx = \Big/_{-2}^{2}\frac{x^4}{4}-\frac{2x^3}{3}+\frac{x^2}{2}$

$= \frac{2^4}{4}-\frac{2}{3}\cdot 2^3+\frac{2^2}{2}-\left(\frac{(-2)^4}{4}-\frac{2}{3}(-2)^3+\frac{(-2)^2}{2}\right)$

$= \cancel{4}-\frac{16}{3}+\cancel{2}-\cancel{4}-\frac{16}{3}-\cancel{2} = -\frac{32}{3}$

d) $\int_0^3 x|x-2|\,dx$

$$|x-2| = \begin{cases} x-2, & \text{kun } x \geq 2 \\ -x+2, & \text{kun } x < 2 \end{cases}$$

$$\left.\begin{aligned} \lim_{x \to 2^-} (-x+2) &= 0 \\ \lim_{x \to 2^+} (x-2) &= 0 = f(2) \end{aligned}\right\} \text{samat} \Rightarrow \text{JATKUVA!}$$

Nyt $x|x-2| = \begin{cases} x(x-2), & \text{kun } x \geq 2 \\ x(-x+2), & \text{kun } x < 2 \end{cases}$

$$= \begin{cases} x^2-2x, & \text{kun } x \geq 2 \\ -x^2+2x, & \text{kun } x < 2 \end{cases}$$

$$\int_0^3 x|x-2|\,dx = \int_0^2 (-x^2+2x)\,dx + \int_2^3 (x^2-2x)\,dx$$

$$= \Big/_0^2 \left(-\frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}}\right) + \Big/_2^3 \left(\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}\right)$$

$$= \frac{-2^3}{3} + 2^2 - \left(-\frac{0^3}{3} + 0^2\right) + \frac{\cancel{3^3}}{\cancel{3}} - \cancel{3^2} - \left(\frac{2^3}{3} - 2^2\right)$$

$$= -\frac{8}{3} + 4 - \frac{8}{3} + 4 = \frac{8}{3}$$

29 a) $\int_1^e \ln x\,dx = \int_1^e 1 \cdot \ln x$

$\left\| \begin{aligned} g(x) &= \ln x \\ g'(x) &= \frac{1}{x} \end{aligned} \qquad \begin{aligned} f'(x) &= 1 \\ f(x) &= x \end{aligned}\right.$

$$= \Big/_1^e x\ln x - \int_1^e x \cdot \frac{1}{x}\,dx$$

$$= e\ln e - 1 \cdot \ln 1 - \int_1^e 1\,dx$$

$$= e - \Big/_1^e x = e - (e-1) = 1$$

b) $\displaystyle\int_1^2 \frac{x+3}{x^2+3x+2}\,dx$ $\qquad \| D(x^2+3x+2) = 2x+3$

ei voida käyttää $\int \frac{f'}{f}\,dx$

Jaetaan nimittäjä tekijöihin

$$x^2+3x+2=0 \quad \rightarrow x = \frac{-3 \pm \sqrt{3^2-4\cdot1\cdot2}}{2\cdot1} = \frac{-3\pm1}{2} = \begin{cases} -1 \\ -2 \end{cases}$$

$$\Rightarrow x^2+3x+2 = (x+1)(x+2)$$

Osamurtokehitelmä: $\displaystyle\frac{x+3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2} \qquad \| (x+1)(x+2)$

$$\begin{aligned} \Rightarrow \quad x+3 &= A(x+2)+B(x+1) \\ &= Ax+2A+Bx+B \\ &= (A+B)x+(2A+B) \end{aligned}$$

$$\Leftrightarrow \quad - \begin{cases} A+B=1 \\ 2A+B=3 \end{cases}$$

$$-A=-2 \quad \Rightarrow A=2$$

$$\Rightarrow B = 1-A = 1-2 = -1$$

$$\Rightarrow \frac{x+3}{(x+1)(x+2)} = \frac{2}{x+1} + \frac{-1}{x+2}$$

$$\Rightarrow \int_1^2 \frac{x+3}{(x+1)(x+2)}\,dx = \int_1^2 \frac{2}{x+1}\,dx - \int_1^2 \frac{1}{x+2}\,dx$$

$$= 2 \Big/_1^2 \ln|x+1| - \Big/_1^2 \ln|x+2|$$

$$= 2(\ln 3 - \ln 2) - (\ln 4 - \ln 3)$$

$$= 2\ln 3 - \underbrace{2\ln 2}_{=\ln 4} - \ln 4 + \ln 3$$

$$= 3\ln 3 - 2\ln 4$$

$$= \ln 27 - \ln 16 = \ln \frac{27}{16}$$