MPTT 2, vko 18

(43) $(1+x^2)\frac{dy}{dx} + x(1+y) = 0$ Separoituva

$\Leftrightarrow (1+x^2)\frac{dy}{dx} = -x(1+y) \quad \| :(1+x^2) \quad \| :(1+y) \neq 0 \quad \| \cdot dx$

$\Leftrightarrow \frac{dy}{1+y} = \frac{-x}{1+x^2}\, dx \quad \| \int$

$\Leftrightarrow \int \frac{1}{1+y}\, dy = -\frac{1}{2}\int \frac{2x}{1+x^2}\, dx$

$\Leftrightarrow \ln|1+y| = -\frac{1}{2}\ln|\underbrace{1+x^2}_{\geq 0}| + C$

$\Leftrightarrow \ln|1+y| = \ln(1+x^2)^{-\frac{1}{2}} + C \quad \| e^{()}$

$\Leftrightarrow e^{\ln|1+y|} = e^{\ln(1+x^2)^{-\frac{1}{2}} + C}$

$\Leftrightarrow e^{\ln|1+y|} = e^{\ln(1+x^2)^{-\frac{1}{2}}} \cdot \underbrace{e^C}_{|C|}$

$\Leftrightarrow |1+y| = (1+x^2)^{-\frac{1}{2}} \cdot |C|$

$\Leftrightarrow 1+y = \pm(1+x^2)^{-\frac{1}{2}} \cdot |C|$ C voi olla positiivinen, negatiivinen tai nolla

$\Leftrightarrow 1+y = \frac{C}{\sqrt{1+x^2}}, \quad C \in \mathbb{R}$

$\Leftrightarrow y = \frac{C}{\sqrt{1+x^2}} - 1, \quad C \in \mathbb{R}$ yleinen ratkaisu

Tarkastellaan tilanne $1+y = 0 \Rightarrow$ $y = -1$ on eräs ratkaisu

$y = -1 \Rightarrow y' = 0 \Rightarrow (1+x^2)\cdot 0 + x(1-1) = 0$

$0 = 0$

Nyt $y = -1$ toteuttaa alkuperäisen diff. yhtälön ja sisältyy yleiseen ratkaisuun, kun $C = 0$.

(41) $\frac{dy}{dx} + \underbrace{2}_{p(x)}\underbrace{y}_{y(x)} = \underbrace{\sin 2x}_{q(x)}$ Täydellinen 1. kertaluvun differentiaaliyhtälö

$\int p(x)\,dx = \int 2\,dx = 2x$

Kerrotaan puolittain lausekkeella $e^{\int p(x)dx} = e^{2x}$

$\Rightarrow \frac{dy}{dx} \cdot e^{2x} + 2ye^{2x} = \sin 2x e^{2x}$

$\Leftrightarrow \frac{d}{dx}(e^{2x} \cdot y) = \sin 2x e^{2x} \quad || \int$

tulon derivoimissääntö

$\Leftrightarrow e^{2x}y = \int \sin 2x e^{2x}\,dx \quad (*)$

Os. int. $g = e^{2x}$ $g' = 2e^{2x}$
$f' = \sin 2x$
$f = \frac{1}{2}\int \sin 2x \cdot 2\,dx$
$= -\frac{1}{2}\cos 2x$

$\Leftrightarrow e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} - \int -\frac{1}{2}\cos 2x \cdot 2e^{2x}\,dx$

$\Leftrightarrow e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} + \int \cos 2x\, e^{2x}\,dx$

$g = e^{2x}$ $g' = 2e^{2x}$
$f' = \cos 2x$
$f = \frac{1}{2}\int 2\cos 2x\,dx$
$= \frac{1}{2}\sin 2x$

$\Leftrightarrow e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} + \frac{1}{2}\sin 2x e^{2x} -$
$- \int \frac{1}{2}\sin 2x \cdot 2e^{2x}\,dx$

$\Leftrightarrow e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} + \frac{1}{2}\sin 2x e^{2x} - \int \sin 2x e^{2x}\,dx$

$\Leftrightarrow \int \sin 2x e^{2x}\,dx = -\frac{1}{2}\cos 2x\, e^{2x} + \frac{1}{2}\sin 2x e^{2x} - e^{2x}y + C$ sij. (*)

$\Rightarrow e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} + \frac{1}{2}\sin 2x e^{2x} - e^{2x}y + C$

$\Leftrightarrow 2e^{2x}y = -\frac{1}{2}\cos 2x e^{2x} + \frac{1}{2}\sin 2x e^{2x} + C \quad || : 2e^{2x} \neq 0$

$\Leftrightarrow y = -\frac{1}{4}\cos 2x + \frac{1}{4}\sin 2x + \frac{C}{e^{2x}}$

$\Leftrightarrow y = -\frac{1}{4}\cos 2x + \frac{1}{4}\sin 2x + Ce^{-2x}$, $C \in \mathbb{R}$ yleinen ratkaisu

(42) $\frac{dy}{dx} + \underbrace{2}_{p(x)}\underbrace{y}_{y(x)} = \underbrace{x^2 + e^x}_{q(x)}$ Täydellinen 1. kertaluvun differentiaaliyhtälö

$\int p(x)\,dx = \int 2\,dx = 2x$

Kerrotaan puolittain lausekkeella $e^{\int p(x)dx} = e^{2x}$

$\Rightarrow \quad \frac{dy}{dx} \cdot e^{2x} + 2ye^{2x} = x^2e^{2x} + e^{3x}$ tulon derivoimissääntö

$\Leftrightarrow \quad \frac{d}{dx}(e^{2x}y) = x^2e^{2x} + e^{3x} \quad \| \int$

$\Leftrightarrow \quad e^{2x}y = \int x^2e^{2x} + e^{3x}\,dx$

$\Leftrightarrow \quad e^{2x}y = \int x^2e^{2x}\,dx + \frac{1}{3}\int 3e^{3x}\,dx$

$\Leftrightarrow \quad e^{2x}y = \frac{1}{3}e^{3x} + \int x^2e^{2x}\,dx \quad (*)$

$\int x^2e^{2x}\,dx$ $\quad g = x^2 \quad f' = e^{2x}$

$= \frac{1}{2}e^{2x}\cdot x^2 - \int \frac{1}{2}e^{2x}\cdot 2x\,dx$ $\quad g' = 2x \quad f = \frac{1}{2}\int 2e^{2x}dx = \frac{1}{2}e^{2x}$

$= \frac{1}{2}e^{2x}\cdot x^2 - \int xe^{2x}\,dx$

$= \frac{1}{2}e^{2x}\cdot x^2 - (\frac{1}{2}e^{2x}\cdot x - \int \frac{1}{2}e^{2x}\cdot 1\,dx)$ $\quad g = x \quad f' = e^{2x}$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\cdot\frac{1}{2}e^{2x} + C$ $\quad g' = 1 \quad f = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$ sij (*):een

$\Rightarrow e^{2x}y = \frac{1}{3}e^{3x} + \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C \quad \| : e^{2x} \neq 0$

$\Leftrightarrow \quad y = \frac{1}{3}e^x + \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + \frac{C}{e^{2x}}$

$\Leftrightarrow \quad y = \frac{1}{3}e^x + \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4} + Ce^{-2x}, \quad C \in \mathbb{R}$

(44) $\frac{dy}{dx} + \underbrace{2x}_{p(x)}\underbrace{y}_{y(x)} = \underbrace{2xe^{-x^2}}_{q(x)}$ Täydellinen 1. kertaluvun differentiaaliyhtälö

$\int p(x)\,dx = \int 2x\,dx = \frac{1}{2}2x^2 = x^2$

Kerrotaan puolittain lausekkeella $e^{\int p(x)dx} = e^{x^2}$

$\Rightarrow \frac{dy}{dx}e^{x^2} + 2xye^{x^2} = 2xe^{-x^2}e^{x^2}$

tulon derivoimissääntö

$\Leftrightarrow \frac{d}{dx}(e^{x^2}y) = 2x \quad || \int$

$\Leftrightarrow e^{x^2}y = \int 2x\,dx$

$\Leftrightarrow e^{x^2}y = x^2 + C \quad || : e^{x^2} \neq 0$

$\Leftrightarrow y = x^2e^{-x^2} + Ce^{-x^2}, \quad C \in \mathbb{R}$