45 a) $(x+y)\,dx + (x-y)\,dy = 0$ alkuehdolla $y(0) = 0$

HOMOGEENINEN. Saatetaan differentiaaliyhtälö normaalimuotoon.

$$(x+y)\,dx + (x-y)\,dy = 0 \qquad \| : dx$$

$$\Leftrightarrow \quad x + y + (x-y)\frac{dy}{dx} = 0 \qquad (*)$$

$$\Leftrightarrow \quad (y-x)\frac{dy}{dx} = x + y \qquad \| : (y-x) \neq 0$$

$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{x+y}{y-x} \qquad (**)$$

Siis $f(x,y) = \dfrac{x+y}{y-x}$

Tutkitaan, onko yhtälö homogeeniyhtälö.
Olkoon $\lambda \neq 0$ mielivaltainen.

Nyt $f(\lambda x, \lambda y) = \dfrac{\lambda x + \lambda y}{\lambda y - \lambda x} = \dfrac{\cancel{\lambda}(x+y)}{\cancel{\lambda}(y-x)} = \dfrac{x+y}{y-x} = f(x,y)$

Täten $(**)$ on homogeeninen differentiaaliyhtälö.

Sij. $u = y/x \Leftrightarrow y = ux \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx}\cdot x + u$

$$(**) \Rightarrow \frac{du}{dx}\cdot x + u = \frac{x+ux}{ux-x} = \frac{\cancel{x}(1+u)}{\cancel{x}(u-1)} = \frac{1+u}{u-1}$$

$$\Leftrightarrow \quad \frac{du}{dx}\cdot x = \frac{1+u}{u-1} - \overset{u-1}{u} = \frac{1+u-u^2+u}{u-1} = \frac{-u^2+2u+1}{u-1} \qquad \| \cdot (u-1) : (-u^2+2u+1) \neq 0 \quad \| \cdot dx \quad \| : x \neq 0$$

$$\Leftrightarrow \quad \frac{u-1}{-u^2+2u+1}\,du = \frac{1}{x}\,dx \qquad \| \int \qquad \text{SEPAROITUVA}$$

$$\Leftrightarrow \quad -\frac{1}{2}\int \frac{-2(u-1)}{-u^2+2u+1}\,du = \int \frac{1}{x}\,dx \qquad \| D(-u^2+2u+1) = -2u+2 = -2(u-1)$$

$\Leftrightarrow \; -\frac{1}{2}\ln|-u^2+2u+1| = \ln|x| + c \quad \| \cdot (-2)$

$\Leftrightarrow \; \ln|-u^2+2u+1| = -2\ln|x| + c \quad \| \, e^{(\,)}$

$\Leftrightarrow \; e^{\ln|-u^2+2u+1|} = e^{\ln|x|^{-2} + c}$

$= e^{\ln|x|^{-2}} \cdot \underbrace{e^c}_{=|C|} \quad \bigstar$

$\Leftrightarrow \; |-u^2+2u+1| = |x|^{-2}\,|C| = \frac{|C|}{|x|^2} = \frac{|C|}{x^2}$

$\Leftrightarrow \; -u^2+2u+1 = \frac{C}{x^2}$ Sij. takaisin $u = \frac{y}{x}$

$\Leftrightarrow \; -\frac{y^2}{x^2} + \frac{2y}{x} + 1 = \frac{C}{x^2} \quad \| \cdot x^2$

$\Leftrightarrow \; -y^2 + 2xy + x^2 = C$

Ratkaistaan C alkuehdon $y(0)=0$ avulla.
Sij. $x=0$ ja $y=0$.

$\Rightarrow \; -0^2 + 2\cdot 0 \cdot 0 + 0^2 = C$

$\Leftrightarrow \; C = 0 \;$ ↯ Piti olla $C = e^c > 0 \; \bigstar$

$\Rightarrow$ ei ratkaisua

Tutkitaan nyt aiemmin poissuljetut tapaukset

i) $y - x = 0 \Leftrightarrow y = x \Rightarrow \frac{dy}{dx} = 1$

$(*) \Rightarrow \; x + x + (x-x)\cdot 1 = 0$

$\Leftrightarrow \; 2x = 0$

Pätee vain, kun $x=0$, siis $y=x$ ei ole yleinen ratkaisu

ii) $-u^2 + 2u + 1 = 0 \Leftrightarrow -\frac{y^2}{x^2} + \frac{2y}{x} + 1 = 0 \quad \| \cdot (-x^2)$

$\Leftrightarrow \; y^2 + 2xy - x^2 = 0$

$$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4\cdot 1 \cdot (-x^2)}}{2\cdot 1} = \frac{2x \pm \sqrt{4x^2 + 4x^2}}{2}$$

$$= \frac{2x \pm \sqrt{8x^2}}{2} = \frac{2x \pm 2x\sqrt{2}}{2} = x \pm x\sqrt{2} = x(1 \pm \sqrt{2})$$

$$(**) \Rightarrow 1+\sqrt{2} = \frac{x + x(1+\sqrt{2})}{x(1+\sqrt{2}) - x} = \frac{\cancel{x}(2+\sqrt{2})}{\cancel{x}\sqrt{2}} = \frac{2+\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\cancel{\sqrt{2}}\sqrt{2} + \cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} = \sqrt{2} + 1 \quad \text{ok}$$

Vastaavasti, kun $y = (1-\sqrt{2})x$.

Alkuehto $y(0) = 0 \Rightarrow y = 0(1 \pm \sqrt{2}) = 0$ ok.

Nähdään, että $y = (1 \pm \sqrt{2})x$ ovat differentiaaliyhtälön ratkaisuja ja lisäksi ne toteuttavat alkuehdon.

Vast. $y = (1+\sqrt{2})x$ tai $y = (1-\sqrt{2})x$

b) $(x+y)\,dx + (x-y)\,dy = 0$ $(*)$ alkuehdolla $y(0)=0$

EKSAKTI. Nyt $M(x,y) = x+y$ ja $N(x,y) = x-y$

$$\frac{\partial M(x,y)}{\partial y} = 1 \quad \text{ja} \quad \frac{\partial N(x,y)}{\partial x} = 1. \text{ Siis } \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

$\Rightarrow (*)$ on eksakti differentiaaliyhtälö.

Etsitään $g(x,y)$, jolla $\frac{\partial g}{\partial x} = M$ ja $\frac{\partial g}{\partial y} = N$

$$\begin{cases} g = \int M(x,y)\,dx + \varphi(y) \\ \frac{\partial g}{\partial y} = N(x,y) = x - y \end{cases}$$

$$g = \int x + y\,dx + \varphi = \frac{1}{2}x^2 + yx + \varphi(y)$$

$$\frac{\partial g}{\partial y} = x + \varphi'(y) = x - y \quad \Rightarrow \varphi'(y) = -y$$

$\Rightarrow \varphi(y) = -\frac{1}{2} y^2 + C$

Siis $g(x,y) = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + C$

Tehtävä palautuu homogeenisen diff. yhtälön käsittelyyn a)-kohta.

(takaisin sijoitus $u = y/x$ jälkeinen osa)