

11111 2, vko 19

45) a) $(x+y)dx + (x-y)dy = 0$ alkuehdolla $y(0) = 0$

HOMOGEENINEN. Saatetaan differentiaaliyhtälö normaalimuotoon.

$$(x+y)dx + (x-y)dy = 0 \quad ||: dx$$

$$\Leftrightarrow x+y + (x-y) \frac{dy}{dx} = 0 \quad (*)$$

$$\Leftrightarrow (y-x) \frac{dy}{dx} = x+y \quad ||: (y-x) \neq 0$$

$$\Leftrightarrow \frac{dy}{dx} = \frac{x+y}{y-x} \quad (**)$$

Sis $f(x,y) = \frac{x+y}{y-x}$

Tutkitaan, onko yhtälö homogeeniyhtälö.
Olkoon $\lambda \neq 0$ mielivaltainen.

$$\text{Nyt } f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda y - \lambda x} = \frac{\lambda(x+y)}{\lambda(y-x)} = \frac{x+y}{y-x} = f(x,y)$$

Täten (**) on homogeeninen differentiaaliyhtälö.

$$\text{Sij. } u = y/x \Leftrightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot x + u$$

$$(**) \Rightarrow \frac{du}{dx} \cdot x + u = \frac{x+ux}{ux-x} = \frac{x(1+u)}{x(u-1)} = \frac{1+u}{u-1}$$

$$\Leftrightarrow \frac{du}{dx} \cdot x = \frac{1+u}{u-1} - u = \frac{1+u-u^2+u}{u-1} = \frac{-u^2+2u+1}{u-1} \quad || \cdot (u-1) : (-u^2+2u+1) \neq 0$$

$$\Leftrightarrow \frac{u-1}{-u^2+2u+1} du = \frac{1}{x} dx \quad || \int \text{SEPAROITUVA}$$

$$\Leftrightarrow -\frac{1}{2} \int \frac{-2(u-1)}{-u^2+2u+1} du = \int \frac{1}{x} dx \quad || D(-u^2+2u+1) = -2u+2 = -2(u-1)$$

$$\Leftrightarrow -\frac{1}{2} \ln |-u^2 + 2u + 1| = \ln |x| + c \quad || \cdot (-2)$$

$$\Leftrightarrow \ln |-u^2 + 2u + 1| = -2 \ln |x| + c \quad || e^{(\cdot)}$$

$$\begin{aligned} \Leftrightarrow e^{\ln |-u^2 + 2u + 1|} &= e^{\ln |x|^{-2} + c} \\ &= e^{\ln |x|^{-2}} \cdot \underbrace{e^c}_{= |c|} \quad \star \end{aligned}$$

$$\Leftrightarrow |-u^2 + 2u + 1| = |x|^{-2} |c| = \frac{|c|}{|x|^2} = \frac{|c|}{x^2}$$

$$\Leftrightarrow -u^2 + 2u + 1 = \frac{c}{x^2} \quad \text{Sij. takaisin } u = \frac{y}{x}$$

$$\Leftrightarrow -\frac{y^2}{x^2} + \frac{2y}{x} + 1 = \frac{c}{x^2} \quad || \cdot x^2$$

$$\Leftrightarrow -y^2 + 2xy + x^2 = c$$

Ratkaistaan c alkuehdon $y(0) = 0$ avulla.

Sij. $x = 0$ ja $y = 0$.

$$\Rightarrow -0^2 + 2 \cdot 0 \cdot 0 + 0^2 = c$$

$$\Leftrightarrow c = 0 \quad \nabla \quad \text{Piti olla } c = e^c > 0 \quad \star$$

\Rightarrow ei ratkaisua

Tutkitaan nyt aiemmin poissuljetut tapaukset

$$\text{i) } y - x = 0 \Leftrightarrow y = x \Rightarrow \frac{dy}{dx} = 1$$

$$(\star) \Rightarrow x + x + (x - x) \cdot 1 = 0$$

$$\Leftrightarrow 2x = 0$$

Pätee vain, kun $x = 0$, siis $y = x$ ei ole yleinen ratkaisu

$$\text{ii) } -u^2 + 2u + 1 = 0 \Leftrightarrow -\frac{y^2}{x^2} + \frac{2y}{x} + 1 = 0 \quad || \cdot (-x^2)$$

$$\Leftrightarrow y^2 + 2xy - x^2 = 0$$

$$y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4 \cdot 1 \cdot (-x^2)}}{2} = \frac{2x \pm \sqrt{4x^2 + 4x^2}}{2}$$

$$= \frac{2x \pm \sqrt{8x^2}}{2} = \frac{2x \pm 2x\sqrt{2}}{2} = x \pm x\sqrt{2} = x(1 \pm \sqrt{2})$$

$$(**) \Rightarrow 1 + \sqrt{2} = \frac{x + x(1 + \sqrt{2})}{x(1 + \sqrt{2}) - x} = \frac{x(2 + \sqrt{2})}{x\sqrt{2}} = \frac{2 + \sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}\sqrt{2} + \sqrt{2}}{\sqrt{2}} = \sqrt{2} + 1 \quad \text{ok}$$

Vastaa vasti, kun $y = (1 - \sqrt{2})x$.

Alkuehto $y(0) = 0 \Rightarrow y = 0(1 \pm \sqrt{2}) = 0 \quad \text{ok}$.

Nähdään, että $y = (1 \pm \sqrt{2})x$ ovat differentiaaliyhtälön ratkaisuja ja lisäksi ne toteuttavat alkuehdon.

Vast. $y = (1 + \sqrt{2})x$ tai $y = (1 - \sqrt{2})x$

b) $(x+y)dx + (x-y)dy = 0$ (*) alkuehdolla $y(0) = 0$

EKSAKTI. Nyt $M(x,y) = x+y$ ja $N(x,y) = x-y$

$$\frac{\partial M(x,y)}{\partial y} = 1 \quad \text{ja} \quad \frac{\partial N(x,y)}{\partial x} = 1. \quad \text{Siis} \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

\Rightarrow (*) on eksakti differentiaaliyhtälö.

Etsitään $g(x,y)$, jolla $\frac{\partial g}{\partial x} = M$ ja $\frac{\partial g}{\partial y} = N$

$$\begin{cases} g = \int M(x,y) dx + \varphi(y) \\ \frac{\partial g}{\partial y} = N(x,y) = x-y \end{cases}$$

$$g = \int x+y dx + \varphi = \frac{1}{2}x^2 + yx + \varphi(y)$$

$$\frac{\partial g}{\partial y} = x + \varphi'(y) = x-y \quad \Rightarrow \quad \varphi'(y) = -y$$

$$\Rightarrow \varphi(y) = -\frac{1}{2}y^2 + C$$

$$\text{Siis } g(x,y) = \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 + C$$

Tehtävä palautuu homogeeniseen diff. yhtälön käsitteeseen a)-kohta.

(takaisin riippuvuus $u = y/x$ jälkeinen ora)

$x^2 \frac{dy}{dx} = y^2 - xy$ $\Rightarrow \frac{dy}{dx} = \frac{y^2 - xy}{x^2}$
 $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$
 - tekijä erottaminen x ja y avulla, $u = y/x$ sijoitus
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$x^2 \frac{dy}{dx} = y^2 - xy$ $\Rightarrow \frac{dy}{dx} = \frac{y^2 - xy}{x^2}$
 $\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 - \frac{y}{x}$
 $u = y/x \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = u + x \frac{du}{dx}$

$u + x \frac{du}{dx} = u^2 - u$
 $x \frac{du}{dx} = u^2 - 2u$
 $\frac{du}{u^2 - 2u} = \frac{dx}{x}$
 $\frac{du}{u(u-2)} = \frac{dx}{x}$
 - osittain murtolomien avulla

$\frac{1}{u(u-2)} = \frac{A}{u} + \frac{B}{u-2}$
 $\frac{1}{u(u-2)} = \frac{A(u-2) + B(u)}{u(u-2)}$
 $1 = A(u-2) + B(u)$
 $1 = Au - 2A + Bu$
 $1 = (A+B)u - 2A$
 $A+B = 0$
 $-2A = 1 \Rightarrow A = -\frac{1}{2}$
 $B = \frac{1}{2}$
 $\frac{1}{u(u-2)} = -\frac{1}{2u} + \frac{1}{2(u-2)}$
 $\int \frac{1}{u(u-2)} du = \int \left(-\frac{1}{2u} + \frac{1}{2(u-2)}\right) du$
 $= -\frac{1}{2} \ln|u| + \frac{1}{2} \ln|u-2| + C$
 $= \frac{1}{2} \ln \left| \frac{u-2}{u} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{y/x - 2}{y/x} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{y-x-2y}{y} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{-x-y-2y}{y} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{-x-3y}{y} \right| + C$
 $= \frac{1}{2} \ln \left| \frac{x+3y}{y} \right| + C$