Matematiikan perusteet taloustieteilijöille II

Harjoitusten malliratkaisut, viikko 16, kevät 2014

35. Tehdään Taylorin sarjakehitelmä funktiolle $f(x) = e^{x^2}$ tarkkuudella $k = 3$. Valitaan $a = 1 \in [0, 2]$. Nyt

$$
\begin{aligned}
f(x) &= e^{x^2} & &\Rightarrow & f(1) &= e \\
f'(x) &= 2xe^{x^2} & &\Rightarrow & f'(1) &= 2e \\
f''(x) &= 4x^2e^{x^2} + 2e^{x^2} & &\Rightarrow & f''(1) &= 4e + 2e = 6e \\
f'''(x) &= 8x^3e^{x^2} + 8xe^{x^2} + 4xe^{x^2} & &\Rightarrow & f'''(1) &= 8e + 8e + 4e = 20e.
\end{aligned}
$$

Tällöin

$$f(x) \approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3,$$

joten

$$
\begin{aligned}
\int_0^2 e^{x^2}\,dx &\approx \int_0^2 \left(e + 2e(x-1) + \frac{6e}{2}(x-1)^2 + \frac{20e}{6}(x-1)^3 \right) dx \\
&= e \int_0^2 \left(-1 + 2x + 3(x-1)^2 + \frac{5}{6} \cdot 4(x-1)^3 \right) dx \\
&= e \Big/_0^2 \left(-x + x^2 + (x-1)^3 + \frac{5}{6}(x-1)^4 \right) \\
&= e \left(\left(-2 + 2^2 + (2-1)^3 + \frac{5}{6}(2-1)^4 \right) - \left((-1)^3 + \frac{5}{6}(-1)^4 \right) \right) \\
&= e(-2 + 4 + 1 + \frac{5}{6} + 1 - \frac{5}{6}) = 4e.
\end{aligned}
$$

37. Valitaan $a = 0$. (Huomaa, että muutkin valinnat ovat mahdollisia!)Nyt

$$\begin{aligned}
f(x) &= 2x^2 + 2x + 2 && \Rightarrow && f(0) = 2 \\
f'(x) &= 4x + 2 && \Rightarrow && f'(0) = 2 \\
f''(x) &= 4 && \Rightarrow && f''(0) = 4 \\
f^{(k)}(x) &= 0 \text{ kaikilla } k \geq 3 && \Rightarrow && f^{(k)}(0) = 0 \text{ kaikilla } k \geq 3.
\end{aligned}$$

Tällöin

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\
&= 2 + 2x + 2x^2.
\end{aligned}$$

38. a)

$$\bar{a} = 1 - 2i$$

$$a + b = 1 + 2i + 1 - 3i = 2 - i$$

$$a \cdot b = (1+2i)(1-3i) = 1 - 3i + 2i - 6i^2 = 7 - i$$

$$\frac{a}{b} = \frac{1+2i}{1-3i} = \frac{(1+2i)(1+3i)}{(1-3i)(1+3i)} = \frac{1+3i+2i-6}{1^2+3^2} = \frac{-5+5i}{10} = -\frac{1}{2} + \frac{1}{2}i$$

$$|b| = \sqrt{1^2 + (-3)^2} = \sqrt{10}$$

b) Nyt

$$2x^3 - 2x^2 + 18x - 18 = 0 \quad \Leftrightarrow \quad x^3 - x^2 + 9x - 9 = 0.$$

Merkitään $p(x) = x^3 - x^2 + 9x - 9$. Polynomin $p(x)$ mahdolliset rationaaliset nollakohdat ovat ± 1, ± 3 ja ± 9. Nyt $p(1) = 1^3 - 1^2 + 9 \cdot 1 - 9 = 0$, joten $x - 1$ on polynomin $p(x)$ tekijä. Jakamalla $p(x)$ jakokulmassa tekijällä $x - 1$ saadaan

$$p(x) = x^3 - x^2 + 9x - 9 = (x-1)(x^2+9).$$

Edelleen

$$\begin{aligned} x^2+9=0 \quad &\Leftrightarrow \quad x^2=-9 \quad \Leftrightarrow \quad x=\pm\sqrt{-9} \\ \Leftrightarrow \quad & x=\pm\sqrt{9i^2}=\pm\sqrt{(3i)^2}=\pm 3i. \end{aligned}$$

Siispä

$$2x^3-2x^2+18x-18=0 \quad \Leftrightarrow \quad x=1 \text{ tai } x=3i \text{ tai } x=-3i.$$

41. a)

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos 3x\,dx &= \frac{1}{3}\int_0^{\frac{\pi}{2}} 3\cos 3x\,dx = \frac{1}{3}\Big/_0^{\frac{\pi}{2}} \sin 3x = \frac{1}{3}(\sin\frac{3\pi}{2}-\sin 0) \\ &= \frac{1}{3}(-1-0) = -\frac{1}{3} \end{aligned}$$

b)

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + c$$

c)

$$\begin{aligned} & \cos 2x = 1-2\sin^2 x \quad \Leftrightarrow \quad \sin^2 x = \frac{1}{2}(1-\cos 2x) \\ \Rightarrow \quad & \int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx = \frac{1}{2}\int (1-\frac{1}{2}\cdot 2\cos 2x)\,dx \\ & = \frac{1}{2}(x-\frac{1}{2}\sin 2x)+c = \frac{1}{2}x-\frac{1}{4}\sin 2x + c \end{aligned}$$

d)

$$D\cos 3x = -3\sin 3x$$

e)

$$D\tan 2x = \frac{2}{\cos^2(2x)} = 2+2\tan^2(2x)$$

f)

$$D\sin^2(2x) = 2\sin 2x\cdot\cos 2x\cdot 2 = 2\sin 4x$$