## Introduction to Probability Theory II

## Exercise 2, Autumn 2007

1. Let $X$ and $Y$ be independent random variables with means $\mu_{1}$ and $\mu_{2}$, and variances $\sigma_{1}$ and $\sigma_{2}$, respectively. Present, using these, following
a) $\mathrm{E}(a X+b Y)$, where $a$ and $b$ are constants;
b) $\mathrm{D}^{2}(a X+b Y)$, where $a$ and $b$ are constants;
c) $\mathrm{E}\left(\left(\frac{X-Y}{2}\right)^{2}\right)$.
2. Determine the $p$-fractile of random variable $X$ for $p=0.5, p=0.75$ and $p=0.99$, when
a) $X \sim \operatorname{Tas}(0,1)$,
b) $X \sim \operatorname{Exp}(2)$,
c) $X \sim N\left(1 / 2^{1} / 4\right)$.
3. Let $P(A)=p$. Determine the probability generating function of the indicator $1_{A}$ and use this to determine the probability generating function of distribution $\operatorname{Bin}(n, p)$.
4. Let $X$ be a $\mathbb{N}$-valued random variable and $G$ the probability generating function of $X$.
a) Calculate $G(0)$ and $G(1)$.
b) Express the probability that $X$ is even using $G$.
5. Let $X$ and $Y$ be independent random variables. Determine the conditional distribution

$$
P\{X=k \mid X+Y=n\}, \text { for } k=0,1, \ldots, n,
$$

when
a) $X \sim \operatorname{Bin}\left(n_{1}, p\right)$ and $Y \sim \operatorname{Bin}\left(n_{2}, p\right)$,
b) $X \sim \operatorname{Poisson}\left(\lambda_{1}\right)$ and $Y \sim \operatorname{Poisson}\left(\lambda_{2}\right)$,
c) $X, Y \sim \operatorname{Geom}(p)$.
6. (Jenssen's inequality) Suppose that $g$ is differentiable and its derivative is increasing. Show that, if random variables $X$ and $g(X)$ have expected value, then

$$
g(\mathrm{E}(X)) \leq \mathrm{E}(g(X))
$$

