

Introduction to Probability Theory II

Exercise 6, Autumn 2007

1. *Buffon's needle problem* Let the length of a needle be $2k$ units, where $0 < k < a$. The needle is dropped on a paper ruled with parallel lines $2a$ units apart. Calculate the probability that the needle intersects one of the lines.

Hint: Let X be the distance from the centre of the needle to the closest line and Y the acute angle between the needle and the lines. Now X and Y are independent. Furthermore X is uniformly distributed on the interval $]0, a[$ and Y is uniformly distributed on the interval $]0, \pi[$.

2. Assume that the signal arriving from a satellite is $S = X + Y$, where X is result of an observation and Y is the interference. Assume that $X \perp Y$, $X \sim N(\mu, \sigma_1^2)$ and $(Y \sim N(0, \sigma_2^2))$. Find
 - a) $\text{Corr}(S, X)$,
 - b) Distribution of X conditional on $S = s$.

(Hint: Present the random vector (X, S) as an affine transformation of the random vector (U, V) , where $(U, V) \sim N(0, I)$. Use this to find the density function of random vector (X, S)).

3. Let the density function of the random vector (X, Y) be f , when

$$f(x, y) = ce^{-x^2 - 2y^2} \text{ kaikilla } (x, y) \in \mathbb{R}^2,$$

where c is a constant.

- a) Determine the value of c .
- b) Name the distribution.
- c) Find $E(X)$, $E(Y)$ and $\text{Corr}(X, Y)$.
- d) Assume that the distribution of random vector (X, Y) is a 2-dimensional normal distribution with density function

$$f(x, y) = \frac{1}{2\sqrt{2}\pi} \exp\left[-1/8(3x^2 + 2xy + 3y^2 - 14x - 10y + 19)\right] \text{ for every } (x, y) \in \mathbb{R}^2.$$

Find the expected values for X and Y , and their covariance matrix.

Hint: Find vector z_0 and a matrix C such that the argument of exponential function is $-1/2(z - z_0)^T C^{-1}(z - z_0)$, if $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.