Introduction to Probability Theory II

Exercise 2, Autumn 2009

1. Let X and Y be independent random variables with means μ_1 and μ_2, and variances σ_1 and σ_2, respectively. Present, using these, following
   a) $\mathrm{E}(aX + bY)$, where a and b are constants;
   b) $\mathrm{D}^2(aX + bY)$, where a and b are constants;
   c) $\mathrm{E}\left(\left(\frac{X-Y}{2}\right)^2\right)$.
2. Determine the p–fractile of random variable X for $p = 0.5$, $p = 0.75$ and $p = 0.99$, when
   a) $X \sim \mathrm{Tas}(0, 1)$,
   b) $X \sim \mathrm{Exp}(2)$,
   c) $X \sim \mathrm{N}(^1/_2 {}^1/_4)$.
3. Let $P(A) = p$. Determine the probability generating function of the indicator 1_A and use this to determine the probability generating function of distribution $\mathrm{Bin}(n, p)$.
4. Let X be a $\mathbb{N}$–valued random variable and G the probability generating function of X.
   a) Calculate $G(0)$ and $G(1)$.
   b) Express the probability that X is even using G.
5. Let X and Y be independent random variables. Determine the conditional distribution
   $$P\{X = k \mid X + Y = n\}, \text{ for } k = 0,1, \ldots ,n,,$$
   when
   a) $X \sim \mathrm{Bin}(n_1, p)$ and $Y \sim \mathrm{Bin}(n_2, p)$,
   b) $X, Y \sim \mathrm{Geom}(p)$.
6. *(Jenssen's inequality)* Suppose that g is differentiable and its derivative is increasing. Show that, if random variables X and $g(X)$ have expected value, then
   $$g(\mathrm{E}(X)) \le \mathrm{E}(g(X)).$$
   Hint: Prove first following lemma:
   If g' is increasing, then
   $$g'(y)(x - y) \le g(x) - g(y).$$
   for every $x, y \in \mathbb{R}$.