Introduction to Probability Theory II

Exercise 6, Autumn 2009

1. *Buffon's needle problem* Let the length of a needle be $2k$ units, where $0 < k < a$. The needle is dropped on a paper ruled with parallel lines $2a$ units apart. Calculate the probability that the needle intersects one of the lines.
   Hint: Let X be the distance from the centre of the needle to the closest line and Y the acute angle between the needle and the lines. Now X and Y are independent. Furthermore X is uniformly distributed on the interval $]0, a[$ and Y is uniformly distributed on the interval $]0, \pi[$.
2. Assume that the signal arriving from a satellite is $S = X + Y$, where X is result of an observation and Y is the interference. Assume that $X \perp\!\!\!\perp Y$, $X \sim \mathrm{N}(\mu, \sigma_1^2)$ and $(Y \sim \mathrm{N}(0, \sigma_2^2))$. Find
   a) $\mathrm{Corr}(S, X)$,
   b) Distribution of X conditional on $S = s$.
   (Hint: Present the random vector (X, S) as an affine transformation of the random vector (U, V), where $(U, V) \sim \mathrm{N}(0, I)$. Use this to find the density function of random vector (X, S)).
3. Let the density function of random vector (X, Y) be f, where
   $$f(x, y) = ce^{-x^2-2y^2} \text{ for every } (x, y) \in \mathbb{R}^2,$$
   with $c > 0$.
   a) Determine c.
   b) What is this distribution called?
   c) Find $\mathrm{E}(X)$, $\mathrm{E}(Y)$ and $\mathrm{Corr}(X, Y)$.
4. Let the density function of the random vector (X, Y) be f, when
   $$f(x, y) = ce^{-x^2-2y^2} \text{ kaikilla } (x, y) \in \mathbb{R}^2,$$
   where c is a constant.
   a) Determine the value of c.
   b) Name the distribution.
   c) Find $\mathrm{E}(X)$, $\mathrm{E}(Y)$ and $\mathrm{Corr}(X, Y)$.
   d) Assume that the distribution of random vector (X, Y) is a 2–dimensional normal distribution with density function
   $$f(x, y) = \frac{1}{2\sqrt{2}\pi} \exp\left[-{}^1\!/_8(3x^2 + 2xy + 3y^2 - 14x - 10y + 19)\right] \text{ for every } (x, y) \in \mathbb{R}^2.$$
   Find the expected values for X and Y, and their covariance matrix.
   Hint: Find vector z_0 and a matrix C such that the argument of exponential function is $-{}^1\!/_2(z - z_0)^T C^{-1}(z - z_0)$, if $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and $z_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$.
5. Two points are placed on a line segment of length a randomly and independent of each other.
   a) Let $0 < x < a$. Calculate the probability that the distance between points is greater than x.
   b) Calculate expected valua of the distance.