Matematiikan perusteet taloustieteilijöille Ib

Harjoitus 1, syksy 2013

1.
$$f(x) = \begin{cases} 5x-2, & x \leq 1 \\ 3x, & 1 < x < 2 \\ 2x^2-5, & x \geq 2 \end{cases}$$

Funktio $f(x)$ on jatkuva kohdassa x_0, jos

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0).$$

Funktio $f(x)$ on jatkuva, jos se on jatkuva jokaisessa määrittelyjoukkonsa pisteessä.

Nyt $f(x)$ on jatkuva ainakin, kun $x \neq 1$ ja $x \neq 2$, sillä $5x-2$, $3x$ ja $2x^2-5$ ovat polynomifunktioina jatkuvia määrittelyjoukossaan, ts.

$$\lim_{x \to x_0} f(x) = f(x_0) \quad \text{aina, kun } x_0 \neq 1 \text{ ja } x_0 \neq 2\,.$$

Nyt

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (5x-2) = 5 \cdot 1 - 2 = 3 = f(1)$$

ja

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x) = 3 \cdot 1 = 3,$$

joten $\lim_{x \to 1} f(x)$ on olemassa ja $\lim_{x \to 1} f(x) = f(1)$
$\Rightarrow f(x)$ on jatkuva kohdassa $x = 1$.

Nyt

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (3x) = 3 \cdot 5 = 6$$

ja

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (2x^2-5) = 2 \cdot 2^2 - 5 = 3.$$

Toispuoleiset raja-arvot ovat eri suuret, joten $\lim_{x \to 2} f(x)$ ei ole olemassa
$\Rightarrow f(x)$ ei ole jatkuva kohdassa $x = 2$.

Vast. Funktio $f(x)$ ei ole jatkuva kohdassa $x = 2$

2. a)

$$\begin{aligned} D(x + 2\sqrt{x}) &= D(x + 2x^{\frac{1}{2}}) = Dx + D2x^{\frac{1}{2}} = Dx + 2Dx^{\frac{1}{2}} \\ &= 1 + 2 \cdot \frac{1}{2} x^{\frac{1}{2}-1} = 1 + x^{-\frac{1}{2}} = 1 + \frac{1}{\sqrt{x}} \end{aligned}$$

b)

$$\begin{aligned} \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} &= D(x^{-1} + x^{-2} + x^{-3}) = Dx^{-1} + Dx^{-2} + Dx^{-3} \\ &= (-1)x^{-1-1} + (-2)x^{-2-1} + (-3)x^{-3-1} \\ &= -1x^{-2} - 2x^{-3} - 3x^{-4} \\ &= -\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4} \end{aligned}$$

c)

$$\begin{aligned} D(2 - 3x^2)^3 &= 3(2 - 3x^2)^2 \cdot D(2 - 3x^2) \\ &= 3(2 - 3x^2)^2 \cdot (-3 \cdot 2x) \\ &= -18x(2 - 3x^2)^2 \end{aligned}$$

d)

$$\begin{aligned} D(x^3 - 1)(2x^2 + 3) &= D(x^3 - 1) \cdot (2x^2 + 3) + (x^3 - 1) \cdot D(2x^2 + 3) \\ &= 3x^2 \cdot (2x^2 + 3) + (x^3 - 1) \cdot 4x \\ &= 6x^4 + 9x^2 + 4x^4 - 4x \\ &= 10x^4 + 9x^2 - 4x \end{aligned}$$

e)

$$\begin{aligned}
D(x(2x-3)(5x-4)^3) &= D((2x^2-3x)(5x-4)^3) \\
&= D(2x^2-3x)\cdot(5x-4)^3 + (2x^2-3x)\cdot D(5x-4)^3 \\
&= (4x-3)(5x-4)^3 + (2x^2-3x)\cdot 3\cdot(5x-4)^2\cdot 5 \\
&= (4x-3)(5x-4)^3 + 15(2x^2-3x)(5x-4)^2
\end{aligned}$$

f)

$$\begin{aligned}
D\frac{x+1}{x^2-3} &= \frac{D(x+1)\cdot(x^2-3)-(x+1)\cdot D(x^2-3)}{(x^2-3)^2} \\
&= \frac{1\cdot(x^2-3)-(x+1)\cdot 2x}{(x^2-3)^2} \\
&= \frac{x^2-3-2x^2-2x}{(x^2-3)^2} \\
&= \frac{-x^2-2x-3}{(x^2-3)^2}
\end{aligned}$$

g) Tapa 1.

$$\begin{aligned}
D\sqrt{\frac{x-2}{x+3}} &= D\left(\frac{x-2}{x+3}\right)^{\frac{1}{2}} = \frac{1}{2}\left(\frac{x-2}{x+3}\right)^{-\frac{1}{2}}\cdot D\left(\frac{x-2}{x+3}\right) \\
&= \frac{1}{2\sqrt{\frac{x-2}{x+3}}}\cdot\frac{D(x-2)\cdot(x+3)-(x-2)\cdot D(x+3)}{(x+3)^2} \\
&= \frac{\sqrt{x+3}}{2\sqrt{x-2}}\cdot\frac{1\cdot(x+3)-(x-2)\cdot 1}{(x+3)^2} \\
&= \frac{\sqrt{x+3}}{2\sqrt{x-2}}\cdot\frac{x+3-x+2}{(x+3)^2} = \frac{(x+3)^{\frac{1}{2}}}{2\sqrt{x-2}}\cdot\frac{5}{(x+3)^2} \\
&= \frac{5}{2\sqrt{x-2}(x+3)^{\frac{3}{2}}}
\end{aligned}$$

Tapa 2.

$$
\begin{aligned}
D\sqrt{\frac{x-2}{x+3}} &= D\frac{\sqrt{x-2}}{\sqrt{x+3}} = D\frac{(x-2)^{\frac{1}{2}}}{(x+3)^{\frac{1}{2}}} \\
&= \frac{D(x-2)^{\frac{1}{2}} \cdot (x+3)^{\frac{1}{2}} - (x-2)^{\frac{1}{2}} \cdot D(x+3)^{\frac{1}{2}}}{\left((x+3)^{\frac{1}{2}}\right)^2} \\
&= \frac{\frac{1}{2}(x-2)^{-\frac{1}{2}} \cdot 1 \cdot (x+3)^{\frac{1}{2}} - (x-2)^{\frac{1}{2}} \cdot \frac{1}{2} \cdot (x+3)^{-\frac{1}{2}} \cdot 1}{x+3} \\
&= \frac{1}{2}(x-2)^{-\frac{1}{2}}\frac{(x+3)^{\frac{1}{2}} - (x-2)(x+3)^{-\frac{1}{2}}}{x+3} \\
&= \frac{1}{2\sqrt{x-2}} \cdot \frac{x+3-(x-2)}{(x+3)^{\frac{3}{2}}} \\
&= \frac{5}{2\sqrt{x-2}(x+3)^{\frac{3}{2}}}
\end{aligned}
$$

Tapa 3.

$$
\begin{aligned}
D\sqrt{\frac{x-2}{x+3}} &= D\frac{\sqrt{x-2}}{\sqrt{x+3}} = D\frac{(x-2)^{\frac{1}{2}}}{(x+3)^{\frac{1}{2}}} = D\left((x-2)^{\frac{1}{2}}(x+3)^{-\frac{1}{2}}\right) \\
&= D(x-2)^{\frac{1}{2}} \cdot (x+3)^{-\frac{1}{2}} + (x-2)^{\frac{1}{2}} \cdot D(x+3)^{-\frac{1}{2}} \\
&= \tfrac{1}{2}(x-2)^{-\frac{1}{2}} \cdot 1 \cdot (x+3)^{-\frac{1}{2}} + (x-2)^{\frac{1}{2}} \cdot (-\tfrac{1}{2}) \cdot (x+3)^{-\frac{3}{2}} \cdot 1 \\
&=^{x+3)} \frac{1}{2\sqrt{x-2}(x+3)^{\frac{1}{2}}} -^{(x+2)^{\frac{1}{2}})} \frac{(x-2)^{\frac{1}{2}}}{2(x+3)^{\frac{3}{2}}} \\
&= \frac{x+3}{2\sqrt{x-2}(x+3)^{\frac{3}{2}}} - \frac{x-2}{2\sqrt{x-2}(x+3)^{\frac{3}{2}}} \\
&= \frac{5}{2\sqrt{x-2}(x+3)^{\frac{3}{2}}}
\end{aligned}
$$

h)

$$
\begin{aligned}
D\sqrt[3]{x^2\sqrt[3]{x^2}} &= D\sqrt[3]{x^2(x^2)^{\frac{1}{3}}} = D\sqrt[3]{x^2x^{\frac{2}{3}}} = D\sqrt[3]{x^{\frac{8}{3}}} \\
&= D(x^{\frac{8}{3}})^{\frac{1}{3}} = Dx^{\frac{8}{9}} \\
&= \tfrac{8}{9}x^{-\frac{1}{9}}
\end{aligned}
$$

i)

$$
\begin{aligned}
De^{\sqrt{1-x^2}} &= e^{\sqrt{1-x^2}} \cdot D\sqrt{1-x^2} = e^{\sqrt{1-x^2}} \cdot D(1-x^2)^{\frac{1}{2}} \\
&= e^{\sqrt{1-x^2}} \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) \\
&= \frac{-xe^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}
\end{aligned}
$$

j) Nyt

$$
\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)^2 = \frac{1}{2}\cdot 2\ln\frac{1+x}{1-x} = \ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)
$$

$$
\begin{aligned}
\Rightarrow \quad D\left(\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)^2\right) &= D\left(\ln(1+x) - \ln(1-x)\right) \\
&= \frac{1}{1+x}\cdot 1 - \frac{1}{1-x}\cdot(-1) \\
&=^{1-x)} \frac{1}{1+x} +^{1+x)} \frac{1}{1-x} \\
&= \frac{1-x}{1-x^2} + \frac{1+x}{1-x^2} \\
&= \frac{2}{1-x^2}
\end{aligned}
$$

k)

$$
\begin{aligned}
D(2^x x^2) &= D2^x \cdot x^2 + 2^x \cdot Dx^2 = 2^x \ln 2 \cdot x^2 + 2^x \cdot 2x \\
&= 2^x x(x\ln 2 + 2)
\end{aligned}
$$

3. a) Funktion $f(x) = x^2 - 2x - 1$ keskimääräinen muutosnopeus välillä $[x_1, x_2] = [1, 4]$

$$
\begin{aligned}
\frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{4^2 - 2\cdot 4 - 1 - (1^2 - 2\cdot 1 - 1)}{4-1} \\
&= \frac{7-(-2)}{4-1} = \frac{9}{3} = 3.
\end{aligned}
$$

b) Funktion $f(x) = x^2 + 2x - 1$ hetkellinen muutosnopeus kohdassa $x = 2$

$$f'(x) = 2x - 2$$
$$f'(2) = 2 \cdot 2 - 2 = 2.$$

4.

$$\begin{aligned}\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{3x^2 + 2 - (3x_0^2 + 2)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{3x^2 - 3x_0^2}{x - x_0} \\ &= \lim_{x \to x_0} \frac{3(x - x_0)(x + x_0)}{x - x_0} \\ &= \lim_{x \to x_0} 3(x + x_0) \\ &= 6x_0.\end{aligned}$$