Abstract Measure Theory

Exam 12.11.2012

The exam lasts four hours.

1. a) Define a σ-algebra.
b) Let (X, Γ_X, μ) be a measure space, let (Y, Γ_Y) be a measurable space and let $T : X \to Y$ be a measurable map. Define the image measure $T_*\mu$ of μ by setting $T_*\mu(A) = \mu(T^{-1}(A))$ for all $A \in \Gamma_Y$. Show that $T_*\mu$ is a measure on Γ_Y.

2. Let μ be a measure on σ-algebra Γ and let $A_1, A_2, \ldots \in \Gamma$ be such that $A_1 \subset A_2 \subset \cdots$. Prove that

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \lim_{i \to \infty} \mu(A_i).$$

3. Assume that $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that the derivative f' is a Borel map.

4. Let (X, Γ, μ) be a measure space. Define

$$\text{Int}(f) = \inf\{\int g \, d\mu \mid g \text{ is simple and } f \leq g\}$$

for all measurable $f : X \to [0, \infty)$. Show that $\text{Int}(\alpha f) = \alpha \, \text{Int}(f)$ for all $\alpha \geq 0$. Is $\text{Int}(f) = \int f \, d\mu$ for all measurable $f : X \to [0, \infty)$?

5. Let $f : [0,1] \to \mathbb{R}$ be a Borel map. Prove that $\mathcal{L}^2(\text{Graph}(f)) = 0$, where $\text{Graph}(f) = \{(x, y) \in [0,1] \times \mathbb{R} \mid y = f(x)\}$ is the graph of f.