Abstract Measure Theory

Exam 29.10.2012

The exam lasts four hours.

1. a) Define an outer measure.
b) Let μ be an outer measure on a set X and let $A \subset X$. The restriction of the measure μ to the set A is defined by $\mu|_A(B) = \mu(B \cap A)$ for all $B \subset X$. Prove that a μ-measurable set is $\mu|_A$-measurable.

2. Show that in Carathéodory's construction $\mu_\infty = \mu$, if and only if for all $\varepsilon > 0$, for all $E \in \mathcal{K}$ and for all $\delta > 0$ there exist $E_i \in \mathcal{K}_\delta$, $i \in \mathbb{N}$, such that $E \subset \cup_{i=1}^\infty E_i$ and $\sum_{i=1}^\infty \zeta(E_i) \leq \zeta(E) + \varepsilon$. Here $\mathcal{K}$ is a covering class and ζ is a premeasure.

3. Let μ be the counting measure on $X = (0, \infty)$, let Γ be the σ-algebra of μ-measurable sets, let $\nu = \mu|_{\mathbb{N}}$ and let (X, Γ, ν) be a measure space (see question 1). Is the function $f : X \to \mathbb{R}$, $f(x) = \frac{1}{x}\sin(\frac{\pi}{2}x)$ for all $x \in X$, ν-integrable?

4. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that $\mathcal{L}^2(\text{Graph}(f)) = 0$, where $\text{Graph}(f) = \{(x,y) \in [0,1] \times \mathbb{R} \mid y = f(x)\}$ is the graph of f.

5. Let μ and λ be Radon measures on R^n. Show that μ and λ are mutually singular if and only if $D(\mu, \lambda, x) = \infty$ for μ-almost all $x \in \mathbb{R}^n$.