TEST: "Introduction to partial differential equations", 29.01.2007

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1) Solve the initial value problem and determine the values x_1 and x_2 for which it exists:

$$x_2\partial_1 u(x_1,x_2) + x_1\partial_2 u(x_1,x_2) = u(x_1,x_2), \quad u(x_1,1) = x_1.$$

2) Prove that the following locally integrable function:

$$K(x_1,x_2) = -1, \quad x_1 < 0, x_2 > 0, \quad K(x_1,x_2) = 0, \quad \textit{otherwise}$$

is the fundamental solution of the operator $\partial_2\partial_1$ on $\mathbb{R}^2$.

3) Assume that $f(x,t)$ is uniformly continuous and bounded on $\mathbb{R}^{n+1}_+$. Prove then that the solution of the initial value problem

$$u_t(x,t) = \Delta u(x,t) + f(x,t), \quad x \in \mathbb{R}^n, t > 0, \quad u(x,0) = 0, \quad x \in \mathbb{R}^n,$$

is given by

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K_{t-s}(x-y) f(y,s)\, dy\, ds,$$

where $K_t(x)$ is the Gaussian kernel and all equations hold pointwise.

4) Solve by Fourier method the heat conductor problem in a bar of lenth π cm for which diffusivity is 1. Suppose that the ends of the bar are insulated and the initial temperature distribution of the bar is $f(x) = 1 - \sin x$. Find the steady state temperature distribution of the bar.

5) Solve by Fourier method the following boundary value problem with Cauchy data for one-dimensional wave equation:

$$u_{tt}(x,t) = u_{xx}(x,t), \quad 0 < x < \pi, \quad t > 0$$

$$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0$$

$$u(x,0) = 0, \quad u_t(x,0) = \sin x, \quad 0 < x < \pi.$$

6) Solve by Fourier method the following boundary value problem for two-dimensional Laplace equation on the rectangle:

$$\Delta u(x,y) = 0, \quad 0 < x < 2, \quad 0 < y < 1,$$

$$u(x,0) = 0, \quad u(x,1) = 0, \quad u(0,y) = 0, \quad u(2,y) = y(1-y).$$

Can we differentiate the series term by term inside of this rectangle? How many times?