Oulun yliopiston matemaattisten tieteiden laitos

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Merkitse tehtäväpaperiin, suoritatko opintojakson aine- vai syventäviin opintoihin. Aineopintoina suorittavien ei tarvitse laskea tehtävien 2., 4. ja 5. (b)-kohtia.

Huom. *Ei omia muistiinpanoja eikä laskinta.*

1. Neliön sivun pituus (cm) on satunnaismuuttuja, jonka jakauman tiheysfunktio $f : \mathbb{R} \to \mathbb{R}$ määritellään

$$f(x) = 4x^3 e^{-x^4} I_{]0,\infty[}(x).$$

Laske neliön pinta-alan jakauman

(a) odotusarvo.

(b) mediaani.

2. Satunnaismuuttuja X noudattaa binomijakaumaa $\text{Bin}(n, \pi)$, jossa $n \in \mathbb{N}_+$ ja $0 < \pi < 1$. Olkoon $Y = \log[X/(n-X)]$ suhteellisen osuuden X/n empiirinen logit-muunnos.

(a) Laske delta-menetelmällä 1. kertaluvun approksimaatio odotusarvolle $\mathbb{E}(Y)$ ja varianssille $\text{var}(Y)$.

(b) Laske 2. kertaluvun approksimaatio odotusarvolle $\mathbb{E}(Y)$.

(c) Ovatko $\mathbb{E}(Y)$ ja $\text{var}(Y)$ olemassa? Perustele!

3. Satunnaisvektorille $X = (Y, Z)$ pätee, että $Y \sim \text{Tas}(0,1)$ ja ehdolla $Y = y$ on Z:n jakauma $\text{Bin}(n, y)$.

(a) Johda Z:n reunajakauman pistetodennäköisyysfunktio.

(b) Johda ehdollisen odotusarvon $\mathbb{E}(Y|Z)$ lauseke.

4. Olkoon $X_1, X_2, \ldots$ jono toisistaan riippumattomia satunnaismuuttujia siten, että $X_i \sim \text{Poisson}(\mu)$, $i = 1, 2, \ldots$, ja $\mu > 0$. Merkitään $\bar{X}_n = \sum_{i=1}^n X_i/n$, $n = 1, 2, \ldots$

(a) Olkoon $U_n = \sqrt{n}(\bar{X}_n - \mu)$. Mitä jakaumaa kohti jono (U_n) suppenee jakaumaltaan, kun $n \to \infty$? Perustele.

(b) Olkoon $V_n = (\bar{X}_n - \mu)^2/\bar{X}_n$, $n = 1, 2, \ldots$. Mitä jakaumaa kohti jono (V_n) suppenee jakaumaltaan, kun $n \to \infty$? Perustele.

5. Olkoon $X = (X_1, X_2)$ satunnaisvektori, joka noudattaa 2-ulotteista normaalijakaumaa, jossa $\mathbb{E}(X_j) = 0$, $\text{var}(X_j) = 1$, $j = 1, 2$, ja $\text{cor}(X_1, X_2) = \varrho \in]-1, +1[$. Olkoon $Y = (Y_1, Y_2)$, jossa $Y_1 = X_1 + X_2$ ja $Y_2 = X_1 - X_2$.

(a) Johda satunnaisvektorin Y yhteisjakauma. Mitä ovat Y:n koordinaattimuuttujien reunajakaumat?

(b) Olkoon nyt $\varrho = 0$ ja määritellään satunnaismuuttuja $V = \sum_{j=1}^{2}(X_j - Y_1/2)^2$. Johda V:n jakauma.

Liitteet. Yksiulotteisten jakaumien taulukko seuraavilla sivuilla. Moniulotteista normaalijakaumaa odotusarvovektorilla μ ja kovarianssimatriisilla Σ noudattavan satunnaisvektorin $X : \Omega \to \mathbb{R}^p$ tiheysfunktion $f(x)$ ja momenttigeneroivan funktion $M(t)$ lausekkeet ovat

$$f(x) = \frac{1}{(2\pi)^{p/2}\sqrt{\det(\Sigma)}} \exp\left\{-\frac{1}{2}(x-\mu)^{\mathrm{T}}\Sigma^{-1}(x-\mu)\right\}, \quad x \in \mathbb{R}^p$$

$$M(t) = \exp\left(\mu^{\mathrm{T}}t + \frac{1}{2}t^{\mathrm{T}}\Sigma t\right), \quad t \in \mathbb{R}^p$$

Table 1 DISCRETE DISTRIBUTIONS

Name of parametric family of distributions	Discrete density functions $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Discrete uniform	$f(x) = \frac{1}{N} I_{\{1,\ldots,N\}}(x)$	$N = 1, 2, \ldots$	$\frac{N+1}{2}$	$\frac{N^2-1}{12}$	$\mu_3' = \frac{N(N+1)^2}{4}$ $\mu_4' = \frac{(N+1)(2N+1)(3N^2+3N-1)}{30}$	$\sum_{j=1}^{N} \frac{1}{N} e^{jt}$
Bernoulli	$f(x) = p^x q^{1-x} I_{\{0,1\}}(x)$	$0 \le p \le 1$ $(q = 1-p)$	p	pq	$\mu_r' = p$ for all r	$q + pe^t$
Binomial	$f(x) = \binom{n}{x} p^x q^{n-x} I_{\{0,1,\ldots,n\}}(x)$	$0 \le p \le 1$ $n = 1, 2, 3, \ldots$ $(q = 1-p)$	np	npq	$\mu_3 = npq(q-p)$ $\mu_4 = 3n^2p^2q^2 + npq(1-6pq)$	$(q + pe^t)^n$
Hypergeometric	$f(x) = \frac{\binom{K}{x}\binom{M-K}{n-x}}{\binom{M}{n}} I_{\{0,1,\ldots,n\}}(x)$	$M = 1, 2, \ldots$ $K = 0, 1, \ldots, M$ $n = 1, 2, \ldots, M$	$n\frac{K}{M}$	$n\frac{K}{M}\frac{M-K}{M}\frac{M-n}{M-1}$	$\mathscr{E}[X(X-1)\cdots(X-r+1)] = r!\frac{\binom{K}{r}\binom{n}{r}}{\binom{M}{r}}$	not useful
Poisson	$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} I_{\{0,1,\ldots\}}(x)$	$\lambda > 0$	λ	λ	$\kappa_r = \lambda$ for $r = 1, 2, \ldots$ $\mu_3 = \lambda$ $\mu_4 = \lambda + 3\lambda^2$	$\exp[\lambda(e^t - 1)]$
Geometric	$f(x) = pq^x I_{\{0,1,\ldots\}}(x)$	$0 < p \le 1$ $(q = 1-p)$	$\frac{q}{p}$	$\frac{q}{p^2}$	$\mu_3 = \frac{q+q^2}{p^2}$ $\mu_4 = \frac{q+7q^2+q^3}{p^4}$	$\frac{p}{1-qe^t}$
Negative binomial	$f(x) = \binom{r+x-1}{x} p^r q^x I_{\{0,1,\ldots\}}(x)$	$0 < p \le 1$ $r > 0$ $(q = 1-p)$	$\frac{rq}{p}$	$\frac{rq}{p^2}$	$\mu_3 = \frac{r(q+q^2)}{p^3}$ $\mu_4 = \frac{r[q+(3r+4)q^2+q^3]}{p^4}$	$\left(\frac{p}{1-qe^t}\right)^r$

Table 2 CONTINUOUS DISTRIBUTIONS

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu'_r = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Uniform or rectangular	$f(x) = \frac{1}{b-a} I_{[a,b]}(x)$	$-\infty < a < b < \infty$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\mu_r = 0$ for r odd $\mu_r = \frac{(b-a)^r}{2^r(r+1)}$ for r even	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-(x-\mu)^2/2\sigma^2]$	$-\infty < \mu < \infty$ $\sigma > 0$	μ	σ^2	$\mu_r = 0$, r odd; $\mu_r = \frac{r!}{(r/2)!}\frac{\sigma^r}{2^{r/2}}$, r even; $\kappa_r = 0$, $r > 2$	$\exp[\mu t + \frac{1}{2}\sigma^2 t^2]$
Exponential	$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\mu'_r = \frac{\Gamma(r+1)}{\lambda^r}$	$\frac{\lambda}{\lambda - t}$ for $t < \lambda$
Gamma	$f(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x} I_{(0,\infty)}(x)$	$\lambda > 0$ $r > 0$	$\frac{r}{\lambda}$	$\frac{r}{\lambda^2}$	$\mu'_j = \frac{\Gamma(r+j)}{\lambda^j \Gamma(r)}$	$\left(\frac{\lambda}{\lambda - t}\right)^r$ for $t < \lambda$
Beta	$f(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} I_{(0,1)}(x)$	$a > 0$ $b > 0$	$\frac{a}{a+b}$	$\frac{ab}{(a+b+1)(a+b)^2}$	$\mu'_r = \frac{B(r+a,b)}{B(a,b)}$	not useful
Cauchy	$f(x) = \frac{1}{\pi\beta\{1 + [(x-\alpha)/\beta]^2\}}$	$-\infty < \alpha < \infty$ $\beta > 0$	Does not exist	Does not exist	Do not exist	Characteristic function is $e^{i\alpha t - \beta\|t\|}$
Lognormal	$f(x) =$ $\frac{1}{x\sqrt{2\pi}\sigma} \exp[-(\log_e x - \mu)^2/2\sigma^2] I_{(0,\infty)}(x)$	$-\infty < \mu < \infty$ $\sigma > 0$	$\exp[\mu + \frac{1}{2}\sigma^2]$	$\exp[2\mu + 2\sigma^2]$ $-\exp[2\mu + 2\sigma^2]$	$\mu'_r = \exp[r\mu + \frac{1}{2}r^2\sigma^2]$	not useful
Double exponential	$f(x) = \frac{1}{2\beta} \exp\left(-\frac{\|x-\alpha\|}{\beta}\right)$	$-\infty < \alpha < \infty$ $\beta > 0$	α	$2\beta^2$	$\mu_r = 0$ for r odd; $\mu_r = r!\,\beta^r$ for r even	$\frac{e^{\alpha t}}{1 - (\beta t)^2}$

Table 2 CONTINUOUS DISTRIBUTIONS (*continued*)

Name of parametric family of distributions	Cumulative distribution function $F(\cdot)$ or probability density function $f(\cdot)$	Parameter space	Mean $\mu = \mathscr{E}[X]$	Variance $\sigma^2 = \mathscr{E}[(X-\mu)^2]$	Moments $\mu_r' = \mathscr{E}[X^r]$ or $\mu_r = \mathscr{E}[(X-\mu)^r]$ and/or cumulants κ_r	Moment generating function $\mathscr{E}[e^{tX}]$
Weibull	$f(x) = abx^{b-1}\exp[-ax^b]I_{(0,\infty)}(x)$	$a > 0$ $b > 0$	$a^{-1/b}\Gamma(1+b^{-1})$	$a^{-2/b}[\Gamma(1+2b^{-1}) - \Gamma^2(1+b^{-1})]$	$\mu_r' = a^{-r/b}\Gamma\left(1+\frac{r}{b}\right)$	$\mathscr{E}[X^t] = a^{-t/b}\Gamma\left(1+\frac{t}{b}\right)$
Logistic	$F(x) = [1+e^{-(x-\alpha)/\beta}]^{-1}$	$-\infty < \alpha < \infty$ $\beta > 0$	α	$\frac{\beta^2\pi^2}{3}$		$e^{\alpha t}\pi\beta t \csc(\pi\beta t)$
Pareto	$f(x) = \frac{\theta x_0^\theta}{x^{\theta+1}} I_{(x_0,\infty)}(x)$	$x_0 > 0$ $\theta > 0$	$\frac{\theta x_0}{\theta - 1}$ for $\theta > 1$	$\frac{\theta x_0^2}{(\theta-1)^2(\theta-2)}$ for $\theta > 2$	$\mu_r' = \frac{\theta x_0^r}{\theta - r}$ for $\theta > r$	does not exist
Gumbel or extreme value	$F(x) = \exp(-e^{-(x-\alpha)/\beta})$	$-\infty < \alpha < \infty$ $\beta > 0$	$\alpha + \beta\gamma$, $\gamma \approx .577216$	$\frac{\pi^2\beta^2}{6}$	$\kappa_r = (-\beta)^r\psi^{(r-1)}(1)$ for $r \geq 2$, where $\psi(\cdot)$ is digamma function	$e^{\alpha t}\Gamma(1-\beta t)$ for $t < 1/\beta$
t distribution	$f(x) = \frac{\Gamma[(k+1)/2]}{\Gamma(k/2)} \frac{1}{\sqrt{k\pi}} \frac{1}{(1+x^2/k)^{(k+1)/2}}$	$k > 0$	$\mu = 0$ for $k > 1$	$\frac{k}{k-2}$ for $k > 2$	$\mu_r = 0$ for $k > r$ and r odd $\mu_r = \frac{k^{r/2}B((r+1)/2, (k-r)/2)}{B(\frac{1}{2}, k/2)}$ for $k > r$ and r even	does not exist
F distribution	$f(x) = \frac{\Gamma[(m+n)/2]}{\Gamma(m/2)\Gamma(n/2)}\left(\frac{m}{n}\right)^{m/2} \times \frac{x^{(m-2)/2}}{[1+(m/n)x]^{(m+n)/2}} I_{(0,\infty)}(x)$	$m, n = 1, 2, \ldots$	$\frac{n}{n-2}$ for $n > 2$	$\frac{2n^2(m+n-2)}{m(n-2)^2(n-4)}$ for $n > 4$	$\mu_r' = \left(\frac{n}{m}\right)^r \frac{\Gamma(m/2+r)\Gamma(n/2-r)}{\Gamma(m/2)\Gamma(n/2)}$ for $r < \frac{n}{2}$	does not exist
Chi-square distribution	$f(x) = \frac{1}{\Gamma(k/2)}\left(\frac{1}{2}\right)^{k/2} x^{k/2-1}e^{-(1/2)x}I_{(0,\infty)}(x)$	$k = 1, 2, \ldots$	k	$2k$	$\mu_j' = \frac{2^j\Gamma(k/2+j)}{\Gamma(k/2)}$	$\left(\frac{1}{1-2t}\right)^{k/2}$ for $t < 1/2$